OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

- Laplacian eigenfunctions and neural networks:… We calculate the needed value of $\left(\frac{1}{2\pi}\int_\Omega (\text{layer\_densities})^2 da\right)^{1/2}$:

$$\|\cdot\| = \left(\frac{1}{2\pi}\left[\int_{\Omega_0}(n_0/a_0)^2 da + \ldots + \int_{\Omega_3}(n_3/a_3)^2 da\right]\right)^{1/2}$$

$$= \left(\frac{1}{2\pi}\left[\int_{\Omega_0}(32/\pi)^2 da + \int_{\Omega_1}(32/(3\pi))^2 da + \int_{\Omega_2}(32/(5\pi))^2 da + \int_{\Omega_3}(32/(7\pi))^2 da\right]\right)^{1/2}$$

$$= \left(\frac{1}{2\pi}\left[\frac{\pi}{8}\cdot\left(\frac{32}{\pi}\right)^2 + 3\frac{\pi}{8}\left(\frac{32}{3\pi}\right)^2 + 5\frac{\pi}{8}\left(\frac{32}{5\pi}\right)^2 + 7\frac{\pi}{8}\left(\frac{32}{7\pi}\right)^2\right]\right)^{1/2}$$

$$= \left(\frac{1}{16}\left[32^2\cdot\left(\frac{1}{\pi^2} + \frac{1}{3\pi^2} + \frac{1}{5\pi^2} + \frac{1}{7\pi^2}\right)\right]\right)^{1/2} = \ldots$$

$$= \frac{8}{\pi}\left(\frac{3\cdot5\cdot7 + 5\cdot7 + 3\cdot7 + 3\cdot5}{3\cdot5\cdot7}\right)^{1/2} = \frac{8}{\pi}\left(\frac{105+35+21+15}{105}\right)^{1/2}$$

$$= \frac{8}{\pi}\left(\frac{176}{105}\right)^{1/2} = \frac{8}{\pi}\left(\frac{11\cdot16}{105}\right)^{1/2} = \frac{32}{\pi}\sqrt{11/105}.$$

!

Here, Ω_i is layer i, $i=0\ldots3$, over which the associated squared density must be integrated, i.e., $\Omega = \cup_{i=0}^{3}\Omega_i$. Normalization of the original density values n_i/a_i is now achieved by dividing them by $\frac{32}{\pi}\sqrt{11/105}$:

$$\frac{32}{(2i+1)\pi} : \left(\frac{32}{\pi}\sqrt{11/105}\right) = \frac{32}{(2i+1)\pi}\cdot\frac{\pi\sqrt{105}}{32\sqrt{11}} = \frac{1}{2i+1}\sqrt{\frac{105}{11}}.$$

!

The approximate values are provided in the table on the previous page. To determine the value of $\|\cdot\|$ of the probability density function after normalization, one must compute the value of

$$\left(\frac{1}{2\pi}\int_\Omega (\text{normalized\_layer\_densities})^2 da\right)^{1/2}.$$

…

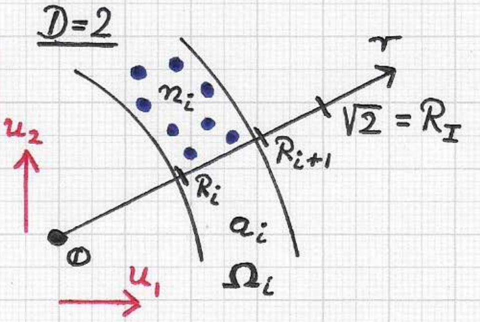
OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd

- Laplacian eigenfunctions and neural networks:... We calculate this value – merely to check whether we have indeed constructed the density function with norm 1:

$$\|\cdot\| = \Big(\tfrac{1}{2\pi}\Big[\int_{\Omega_0}(\sqrt{105/11})^2\,da + \int_{\Omega_1}(\tfrac{1}{3}\sqrt{105/11})^2\,da + \int_{\Omega_2}(\tfrac{1}{5}\sqrt{105/11})^2\,da + \int_{\Omega_3}(\tfrac{1}{7}\sqrt{105/11})^2\,da\Big]\Big)^{1/2}$$

$$= \ldots = \Big(\tfrac{1}{16}\Big[\tfrac{105}{11}\cdot\tfrac{(3\cdot5\cdot7+5\cdot7+3\cdot7+3\cdot5)}{3\cdot5\cdot7}\Big]\Big)^{1/2}$$

$$= \ldots = \Big(\tfrac{1}{16}\cdot\tfrac{176}{11}\Big)^{1/2} = \Big(\tfrac{1}{16}\cdot\tfrac{16\cdot11}{11}\Big)^{1/2} = 1.$$



Layers Ω_i with areas a_i.

Instead of using a uniform, equidistant spacing between all consecutive r-line intervals $[R_0, R_1)$, $[R_1, R_2)$, ..., we can also use arbitrary R_i-values, where $R_0 = 0$, $R_1 = \Delta_0$, $R_2 = R_1 + \Delta_1$, ... The figure above sketches this generalization: A layer is defined by R_i and R_{i+1}, where $\Delta_i = R_{i+1} - R_i$; n_i is the number of points in this layer; and the layer domain Ω_i has the area a_i. For $D=2$, we obtain:

$$\|\cdot\| = \Big(\tfrac{1}{2\pi}\int_{\Omega}(\text{layer\_densities})^2\,da\Big)^{1/2}$$

$$= \Big(\tfrac{1}{2\pi}\sum_{i=0}^{I-1}\int_{\Omega_i}(n_i/a_i)^2\,da\Big)^{1/2} \qquad \{a_i = \pi(R_{i+1}^2 - R_i^2)\}$$

$$= \Big(\tfrac{1}{2\pi}\sum_{i=0}^{I-1}\int_{\Omega_i}\Big(\tfrac{n_i}{\pi(R_{i+1}^2 - R_i^2)}\Big)^2\,da\Big)^{1/2}$$

$$= \ldots$$

...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks: ...

$$= \left(\frac{1}{2\pi} \sum_{i=0}^{I-1} \left(\pi \left(R_{i+1}^2 - R_i^2 \right) \right) \cdot \left(\frac{n_i}{\pi \left(R_{i+1}^2 - R_i^2 \right)} \right)^2 \right)^{1/2}$$

$$= \left(\frac{1}{2\pi} \sum_{i=0}^{I-1} \frac{n_i^2}{\pi \left(R_{i+1}^2 - R_i^2 \right)} \right)^{1/2}$$

$$= \left(\frac{1}{2\pi^2} \sum_{i=0}^{I-1} \frac{n_i^2}{R_{i+1}^2 - R_i^2} \right)^{1/2} = \frac{\sqrt{2}}{2\pi} \left(\sum_{i=0}^{I-1} \frac{n_i^2}{R_{i+1}^2 - R_i^2} \right)^{1/2}.$$

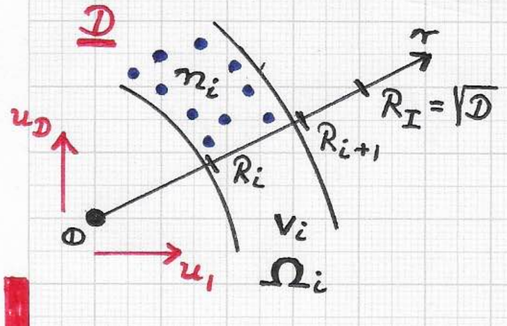
(We can apply this formula to the values of the last example, where $I = 4$, $n_i = 4$ and $R_{i+1} - R_i = \sqrt{2}/4$, $i = 0 \ldots 3$. The values of $R_{i+1}^2 - R_i^2$, where $R_i = i\sqrt{2}/4$, $i = 0 \ldots 4$, are $1/8$, $3/8$, $5/8$ and $7/8$ for $i = 0$, 1, 2 and 3, respectively. Thus, one obtains

$$\|\cdot\| = \left(\sqrt{2}/(2\pi) \right) \cdot \left(16 \cdot \left(8/1 + 8/3 + 8/5 + 8/7 \right) \right)^{1/2} = \ldots$$

$$= \left(\sqrt{2}/(2\pi) \right) \cdot \left(128 \cdot \left(1 + 1/3 + 1/5 + 1/7 \right) \right)^{1/2} = \ldots$$

$$= \left(\sqrt{2}/(2\pi) \right) \cdot \left(128 \cdot \frac{176}{105} \right)^{1/2}$$

$$= \left(\sqrt{2}/(2\pi) \right) \cdot 8\sqrt{2} \cdot 4\sqrt{11/105} = \frac{32}{\pi} \sqrt{\frac{11}{105}}\ .)$$



The most general case, for the D-dimensional setting, is discussed next, see left figure. Here, the "total normalized u-space domain" Ω is a D-ball with radius $\sqrt{D}$, and the individual layer hypervolumes are called v_i, associated with the layer domain Ω_i. Again, n_i denotes the number of points '•' in layer i.

...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... The $\|\cdot\|$-value for the general D-dimensional domain is:

$$\|\cdot\| = \left(\frac{1}{V(D,\Omega)} \int_\Omega (\text{hyper\_layer\_densities})^2 \, dv \right)^{1/2}$$
$$= \left(\frac{1}{V(D,\Omega)} \sum_{i=0}^{I-1} \int_{\Omega_i} (n_i / v_i)^2 \, dv \right)^{1/2}$$
$$= \left(\frac{1}{c_D \cdot D^{D/2}} \sum_{i=0}^{I-1} \int_{\Omega_i} \frac{n_i^2}{(c_D \cdot (R_{i+1}^D - R_i^D))^2} \, dv \right)^{1/2}$$
$$= \left(\frac{1}{c_D \cdot D^{D/2}} \sum_{i=0}^{I-1} \frac{n_i^2}{c_D \cdot (R_{i+1}^D - R_i^D)} \right)^{1/2}$$
$$= \frac{1}{\sqrt{c_D} \cdot D^{D/4}} \cdot \frac{1}{\sqrt{c_D}} \left(\sum_{i=0}^{I-1} \frac{n_i^2}{R_{i+1}^D - R_i^D} \right)^{1/2}$$
$$= \frac{1}{c_D \cdot D^{D/4}} \cdot \left(\sum_{i=0}^{I-1} \frac{n_i^2}{R_{i+1}^D - R_i^D} \right)^{1/2} =: \|\cdot\|_\Omega .$$

Here, $V(D,\Omega)$ is the hypervolume of the D-ball with radius $\sqrt{D}$; v_i is the layer hypervolume of layer domain Ω_i, where the layer hypervolume is the difference between the D-ball with radius R_{i+1} and the D-ball with radius R_i, i.e., $c_D \cdot (R_{i+1}^D - R_i^D)$. The coefficient c_D is given as $c_D = 1/((D/2)!) \cdot \pi^{D/2}$ for $D = 2, 4, 6, \ldots$ and $c_D = 2^{\lceil D/2 \rceil}/(D!!) \cdot \pi^{\lfloor D/2 \rfloor}$ for $D = 1, 3, 5, \ldots$, see p. 14 (6/4/2023). Thus, we must divide a given density function defined over the D-ball with radius $\sqrt{D}$ by the value of $\|\cdot\|_\Omega$, derived above, to obtain the needed normalized probability density function.

(Again, we can consider the values $I = 4$, $n_i = 4$, $D = 2$ and $R_i = i\sqrt{2}/4$, $i = 0 \ldots 4$.

•••

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... When evaluating the general formula for this case, we obtain:

$$\|\cdot\|_\Omega = \frac{1}{\pi\, 2^{1/2}} \left(\sum_{i=0}^{3} \frac{16}{((i+1)\sqrt{2}/4)^2 - (i\sqrt{2}/4)^2} \right)^{1/2}$$

$$= \frac{2\sqrt{2}}{\pi} \left(\sum_{i=0}^{3} \frac{1}{(i+1)^2\, 1/8 - i^2\, 1/8} \right)^{1/2}$$

$$= \frac{2\sqrt{2}}{\pi} \left(\sum_{i=0}^{3} \frac{8}{2i+1} \right)^{1/2}$$

$$= \frac{8}{\pi} \left(\frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} \right)^{1/2}$$

$$= \frac{8}{\pi} \left(\frac{105 + 35 + 21 + 15}{3 \cdot 5 \cdot 7} \right)^{1/2}$$

$$= \frac{8}{\pi} \left(\frac{176}{105} \right)^{1/2} = \frac{8}{\pi} \left(\frac{16 \cdot 11}{105} \right)^{1/2} = \frac{32}{\pi} \sqrt{\frac{11}{105}} \text{ .)}$$

Of special interest is the case of uniform, equidistant spacing of the R_i-values, i.e., the case where $R_i = i\sqrt{D}/I$, $i = 0 \ldots I$. For this important special case, one obtains:

$$\|\cdot\|_\Omega = \frac{1}{c_D \cdot D^{D/4}} \left(\sum_{i=0}^{I-1} \frac{n_i^2}{R_{i+1}^D - R_i^D} \right)^{1/2}$$

$$= \frac{1}{c_D \cdot D^{D/4}} \left(\sum_{i=0}^{I-1} \frac{n_i^2}{((i+1)\sqrt{D}/I)^D - (i\sqrt{D}/I)^D} \right)^{1/2}$$

$$= \frac{1}{c_D \cdot D^{D/4}} \left(\sum_{i=0}^{I-1} \frac{1}{(\sqrt{D}/I)^D} \cdot \frac{n_i^2}{(i+1)^D - i^D} \right)^{1/2}$$

$$= \frac{1}{c_D \cdot D^{D/4}} \cdot \frac{I^{D/2}}{D^{D/4}} \left(\sum_{i=0}^{I-1} \frac{n_i^2}{(i+1)^D - i^D} \right)^{1/2}$$

$$= \frac{1}{c_D} \cdot \left(\frac{I}{D} \right)^{D/2} \left(\sum_{i=0}^{I-1} \frac{n_i^2}{(i+1)^D - i^D} \right)^{1/2} .$$

(When using the values $I = 4$, $n_i = 4$, $D = 2$ and $R_i = i\sqrt{2}/4$, $i = 0 \ldots 4$, we obtain $\|\cdot\|_\Omega = \frac{1}{\pi} \left(\frac{4}{2}\right)^1 \left(\sum_{i=0}^{4} \frac{16}{(i+1)^2 - i^2} \right)^{1/2} = \ldots =$

$= \ldots = \frac{32}{\pi} \sqrt{\frac{11}{105}}$.)

• • •