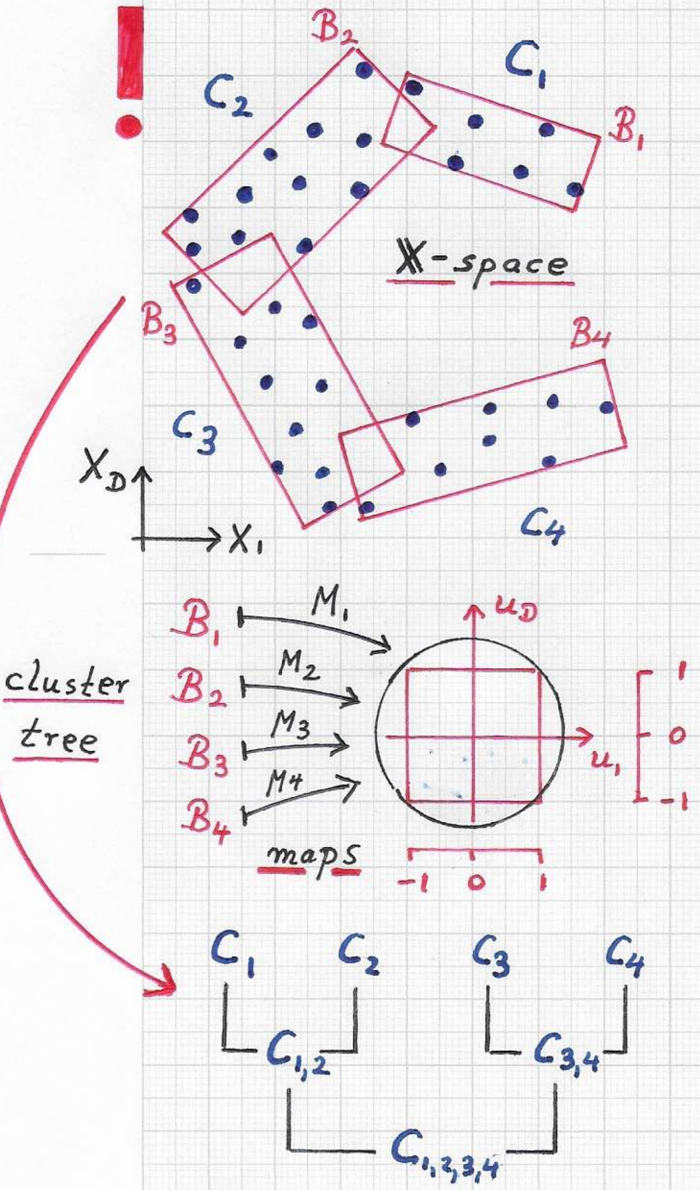
OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:...



We now reconsider the important issue of splitting a point data set, given in D-dimensional $\mathbb{X}$-space, into point clusters, see left figure. The goal is to generate a set of point clusters such that each cluster consists of a "nicely compact subset" of points. A formal definition of "nicely compact point set" is crucially important. Based on this definition, an effective method for splitting must be devised. The (hierarchical) clustering approach sketched in the figure uses a binary tree for the representation of the clusters (cluster hierarchy).

Here, the result is a set of four clusters. Cluster C_i is shown together with its oriented minimal bounding box B_i, $i=1...4$. The cluster-specific linear map M_i is used to map the points in C_i to the normalized $\mathbb{u}$-space domain $[-1,1]^D$.

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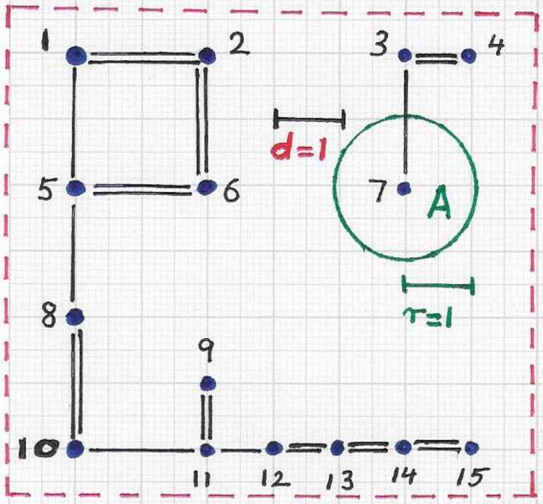
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... It is possible to consider the (one-) nearest-neighbor problem in the context of devising a splitting criterion for a point set. A "tight packing" of points would make splitting unnecessary, while a point set not satisfying a "tight packing" condition would require one to split it. First, one must define and calculate the value of a **SIMPLE** measure to determine whether a given point set satisfies the "tight packing" condition or not. The left figure provides an example in the plane.

B

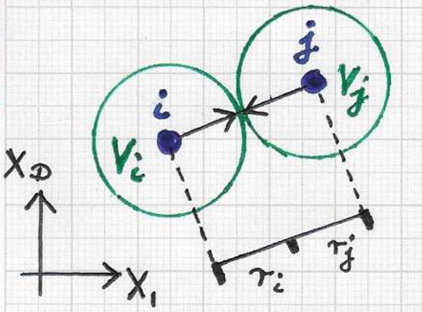


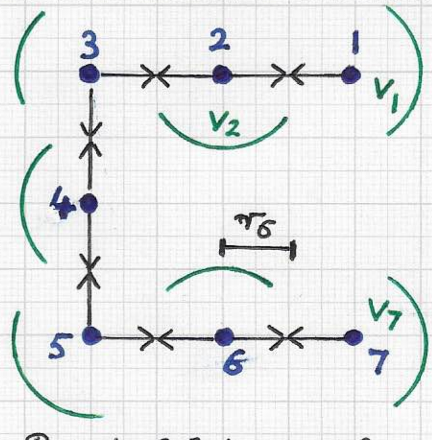
For each point (vertex) **v1**, with $v1 \in \{1,\ldots,15\}$, we determine the vertex (or set of vertices) **v2** that is (are) nearest to **v1**. The figure shows the graph that results when establishing an edge(s) between **v1** and **v2**. (Drawing some edges as "double-edges" merely serves for visualization.) The left table shows for every vertex **v1** the identified vertex (vertices) **v2**, the associated distance **d**, and the disk-area **A** assigned to v1 — where $A = \pi r^2$ and $r = d/2$. ...

v1	v2	d	r	A
1	2,5	2	1	π
2	1,6	2	1	π
3	4	1	1/2	π/4
4	3	1	1/2	π/4
5	1,6,8	2	1	π
6	2,5	2	1	π
7	3	2	1	π
8	5,10	2	1	π
9	11	1	1/2	π/4
10	8,11	2	1	π
11	9,12	1	1/2	π/4
12	11,13	1	1/2	π/4
13	12,14	1	1/2	π/4
14	13,15	1	1/2	π/4
15	14	1	1/2	π/4

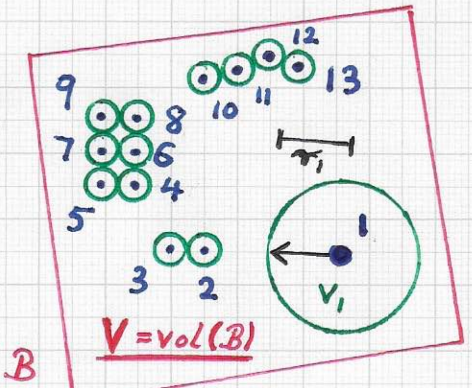
OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks: ...





Points 2, 3, 4, 5 and 6 have two nearest neighbors — each having the same minimal distance value.



Point set with D-balls and a bounding box B. The "bounding primitive" could be a D-box, D-ball or other D-volume.

In this context — point arrangements in $\mathbb{X}$-space — the term "tight packing" is used instead of "high density" to avoid confusion with the discussed concept of a density function. The left figure shows essential parameters: Points i and j are given in D-dimensional $\mathbb{X}$-space; the nearest neighbor of point i (j) is point $j^{(i)}$; the length of the "ray" from i (j) to the midpoint of the directed edge $\overline{ij}$ ($\overline{ji}$) is r_i (r_j); and the hyper-volume of the D-ball with radius r_i (r_j) is called V_i (V_j). The sum $V_i + V_j$ will be used in the measure of tightness. The middle figure shows a 7-point dataset where all radii r_i have the same value; the equal-volume D-balls with hyper-volume V_i are indicated. The bottom figure shows a 13-point structure — where the graph has four unconnected components, given by the point sets $\{1\}$, $\{2,3\}$, $\{4,5,6,7,8,9\}$ and $\{10,11,12,13\}$.

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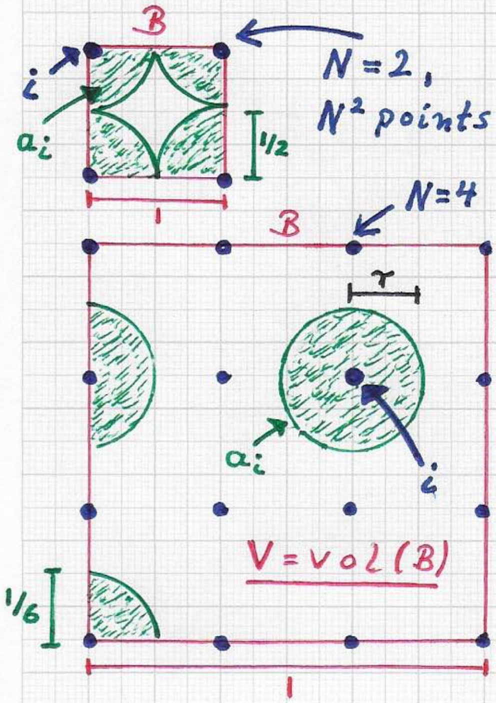
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks: ...



We consider simple cases of point arrangements in the plane ($D=2$) to explain the devised measure of "tightness." The left figure shows the unit (bounding) box B with two sets of "regularly arranged points" of resolution 2×2 in the first case and resolution 4×4 in the second case. We define the measure of "tightness" as follows:

$$T = \left(\Sigma_i\, a_i\right) / V \; .$$

Here, T stands for "tightness," a_i is the hyper-volume (area) of the part of the 2-ball (disk) inside the (hyper-) box B, and V is the area (hyper-volume) of B. In the sketched idealized scenario, all 2-balls have the same radius, i.e., $r = r_i = 1/(2(N-1))$. For $N=2$, we obtain $\Sigma_i\, a_i = 4\cdot 1/4 \cdot \pi \cdot (1/2)^2 = \pi/4$; for $N=4$, we obtain $\Sigma_i\, a_i = 4\cdot 1/4\cdot\pi\cdot(1/6)^2 + 8\cdot 1/2\cdot\pi\cdot(1/6)^2 + 4\cdot\pi\cdot(1/6)^2 = \pi/4$.

Regardless of the value of N, $T = \pi/4$, since $\Sigma_i\, a_i = \pi/4$:

$$\begin{aligned}\Sigma_i\, a_i &= 4\cdot 1/4\cdot \pi\cdot r^2 + 4\cdot(N-2)\cdot 1/2\cdot\pi\cdot r^2 + (N-2)^2\cdot\pi\cdot r^2 \\ &= \pi\cdot r^2\cdot\left(1 + 2N - 4 + N^2 - 4N + 4\right) \\ &= \pi\cdot r^2\cdot\left(N^2 - 2N + 1\right) = \pi\cdot r^2\cdot (N-1)^2 = \pi\,\frac{(N-1)^2}{4(N-1)^2} \\ &= \pi/4. \quad \Rightarrow T = (\pi/4)/V = \pi/4.\end{aligned}$$

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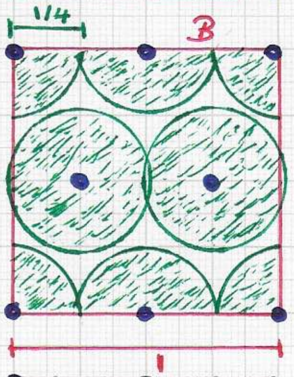
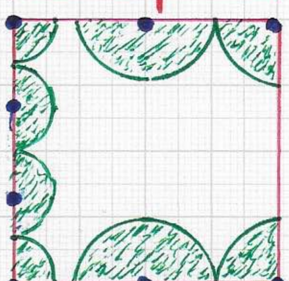
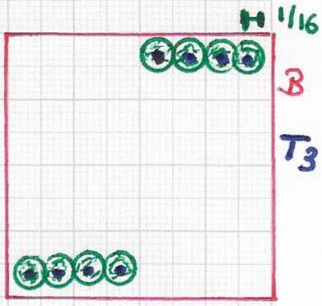
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

- Laplacian eigenfunctions and neural networks:... Despite the fact that this example uses an "ideal" point arrangement (with an underlying uniform, equidistant rectilinear grid structure), it demonstrates the following property of T: When two point sets of different cardinality — containing different points and having different numbers of points in them — consist of points "with the same underlying structure, reflecting the same arrangement," they have the same "tightness" T. (Of course, this statement should be explored in more depth and with mathematical rigor.) We now consider the case where two point sets consist of the same number of points with rather different point arrangements. We compute T-values for the examples shown in the figures on this page:

T_1 

T_2 

 T_3

$$\underline{T_1} = 4\cdot\tfrac{1}{4}\cdot\pi\cdot r^2 + 2\cdot\tfrac{1}{2}\cdot\pi\cdot r^2 + 2\pi\cdot r^2 = 4\pi r^2 = \underline{\pi/4}.$$

$$\underline{T_2} = 2\cdot\tfrac{1}{4}\cdot\pi\cdot(\tfrac{1}{4})^2 + 2\cdot\tfrac{1}{2}\cdot\pi\cdot(\tfrac{1}{4})^2 + 2\cdot\tfrac{1}{4}\cdot\pi\cdot(\tfrac{1}{6})^2 + 2\cdot\tfrac{1}{2}\cdot\pi\cdot(\tfrac{1}{6})^2 = \underline{\tfrac{13}{96}\pi}.$$

$$\underline{T_3} = 8\cdot\pi\cdot r^2 = 8\cdot\pi\cdot(\tfrac{1}{16})^2 = \underline{\pi/32}.$$

These point sets demonstrate that Point sets of the same cardinality but with different point arrangements can have significantly different "tightness" values.

ONE MUST KEEP IN MIND THE INFLUENCE OF B ON T-VALUES.

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