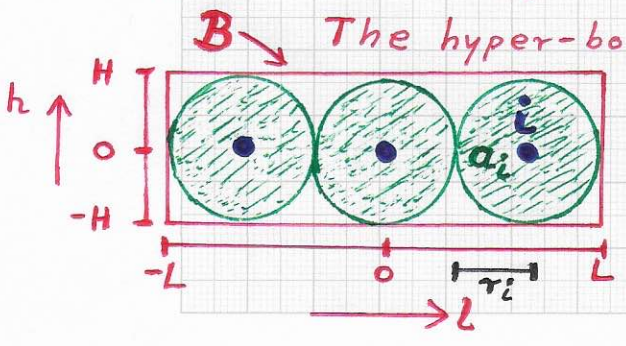
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... We must devise an efficient method for the construction of B – the needed "bounding primitive" enclosing a given point set, e.g., a bounding D-ball, D-dimensional hyper-ellipsoid or D-dimensional hyper-box (-cuboid). B and its hyper-volume V are crucially important to determine a point set's "tightness" T and consequently to split a point set into multiple point subsets with larger T-values. We have the following design goals for B:

i) B should be fundamentally based on the transformation described on pp. 23–24 (4/14–15/2023), concatenating translation, rotation and scaling. B should be the minimal-volume hyper-box, where the box center and box orientation are defined by the associated center and eigen-directions of the point set.

ii) In the context of the introduced "tightness" definition, each point i has an associated D-ball. The hyper-box B should include each D-ball in its interior – to eliminate the need of clipping D-balls and calculating partial D-ball hyper-volumes.



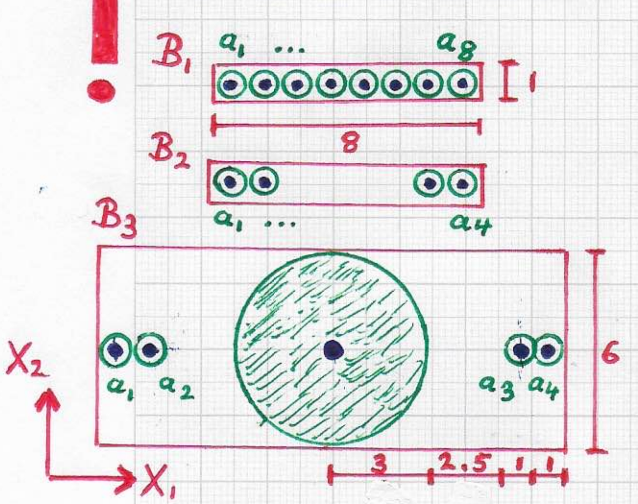
The left figure shows a simple example. The minimal box volume is $V = 4LH$, and $\sum_i a_i = 3\pi r_i^2 = 3\pi r^2$ (since $r_i = r$, $i = 1,2,3$). Thus, $T = \frac{3\pi r^2}{4LH}$.

...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:...

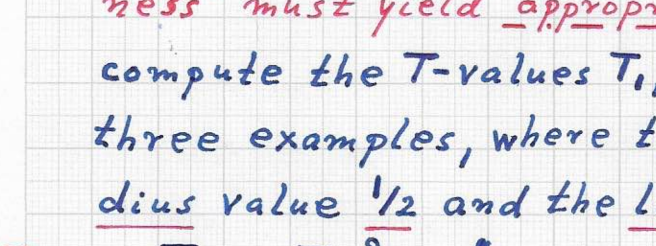
iii) While the given points are embedded in D-dimensional X-space, the statistical eigen-analysis of a point set can reveal that the actual point-set-inherent "eigen-dimensionality" is less than D. The left figure shows three examples of point sets consisting of points on the same line, while the points are embedded in the (X_1, X_2)-plane.



For such a situation, the measure for "tightness" must yield appropriate T-values as well. We compute the T-values T_1, T_2 and T_3 for the sketched three examples, where the small 2-balls have radius value $1/2$ and the large 2-ball has radius value 3:

$$T_1 = \Sigma_{i=1}^{8} a_i \,/\, vol(B_1) = 8\pi(1/2)^2 / 8 = \pi/4;$$
$$T_2 = \Sigma_{i=1}^{4} a_i \,/\, vol(B_2) = 4\pi(1/2)^2 / 8 = \pi/8;$$
$$T_3 = (\pi(3)^2 + \Sigma_{i=1}^{4} a_i) \,/\, vol(B_3) = (9\pi + \pi)/90 = \pi/9.$$

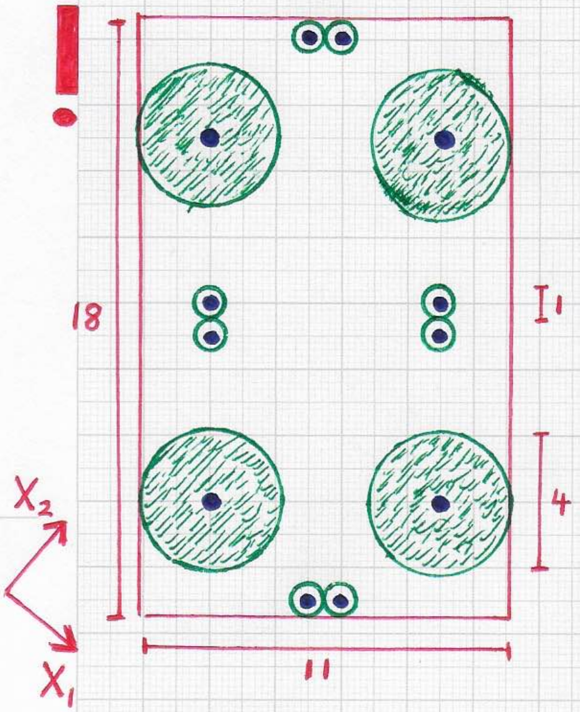
These values imply that the first example represents the tightest packing; that the second example is half as tight as the first example; and that the third example is even less tight than the second example.



The left figure shows equidistantly placed points on a line. In this case, one obtains $T = \pi/4$, regardless of N. ...

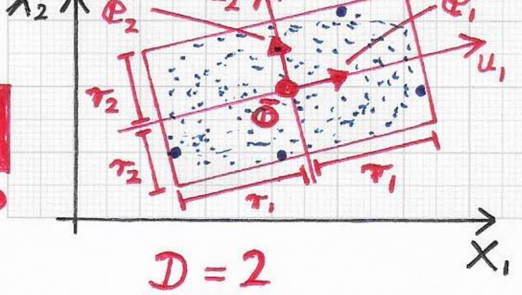
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks: ...



The numerical examples presented demonstrate that the proposed "tightness" measure T produces meaningful, appropriate characterizations of the packings considered. Since T-values will be used to decide whether a point set {•} must be split, we discuss the point set shown in the left figure. The goal is to understand how a low T-value will require one to split a point set – and how splitting is performed and minimal (hyper-) volume bounding (hyper-) boxes are established for the point subsets. The example illustrated in the top-left figure involves 4 points with radius value 2 and 8 points with radius value 1/2. The minimal bounding box has the area $11 \cdot 18 = 198$. Thus, the T-value is

$$\underline{T} = (4\pi\, 2^2 + 8\pi\,(1/2)^2)\,/\,198 = \underline{18\pi\,/\,198}.$$

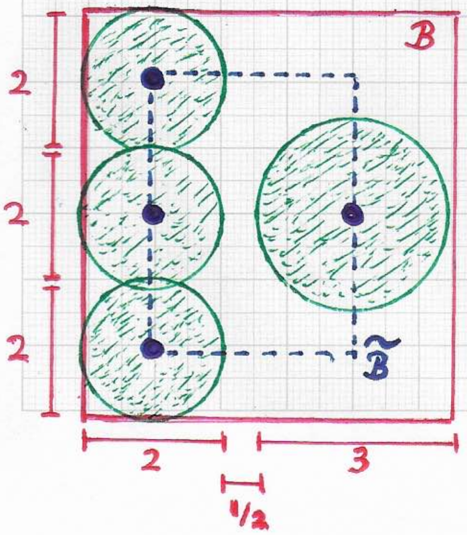


$\underline{D = 2}$

The left figure summarizes the essential parameters defining the X-to-u-space mapping:
– local origin $\overline{O}$,
– orthonormal eigenvectors e_1 and e_2,
– scaling factors r_1 and r_2. ...

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

- Laplacian eigenfunctions and neural networks: ... The $\mathbb{X}$-to-$\mathbb{u}$-space mapping (linear coordinate transformation) is derived and discussed in detail on pp. 16 – 25 (4/9–16/2023). This mapping is based on the concept of determining a minimal bounding (hyper-) box of a point set $\{\bullet\}$, with its orientation defined by the point set's **eigen**directions $(\mathbb{e}_1, \mathbb{e}_2, \ldots, \mathbb{e}_D)$ and its (hyper-) volume defined as $V = 2r_1 \cdot 2r_2 \cdot \ldots \cdot 2r_D = 2^D \cdot \prod_{i=1}^{D} r_i$. For the purpose of calcutating the value of the proposed "tightness" measure T, we must modify the concept of the minimal bounding (hyper-) box: Given a set of points $\{\bullet\}$, where each point has an associated D-ball of a specific radius with the point being the D-ball's center, CONSTRUCT AN ORIENTED MINIMAL (HYPER-) BOX FOR THE SET OF D-BALLS. This modification is necessary since each D-ball must lie entirely in the interior of the box.

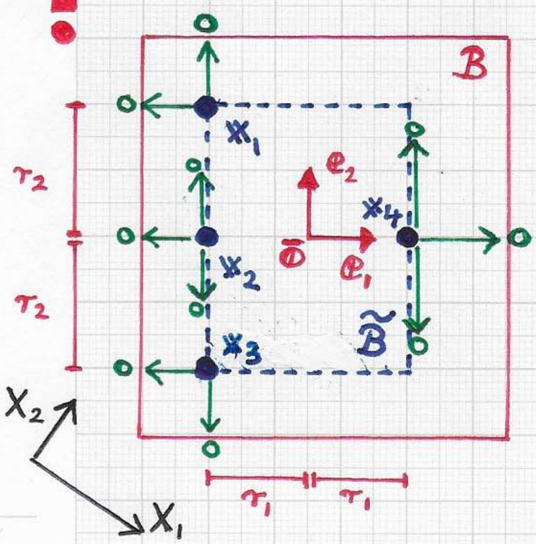


The left figure provides a simple example for $D=2$. The T-value is $T = (3\pi + 9/4\,\pi)/33 = (21/4\,\pi)/33 = 7/44\,\pi$. This value is based on box B, which is the needed modification of the point set's bounding box $\widetilde{B}$.

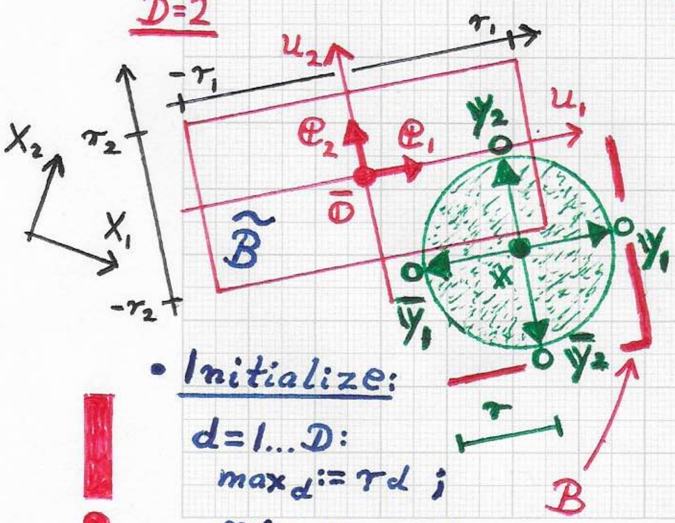
...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks: ...



One can construct the needed bounding box B based on the point set's oriented minimal bounding box $\tilde{B}$. The left figure can serve as a simple example ($D=2$) for a constructive algorithm – for the data and the computational steps involved. Considering the shown example, we are given the X-space point set $\{x_i\}_{i=1}^4$; further, the "eigen-analysis" of the point set generates the local origin $\bar{o}$, the two orthonormal eigen-directions (basis vectors) e_1 and e_2, and the two scaling factors r_1 and r_2 for the point set's minimal bounding box $\tilde{B}$. The left figure shows the essential:



• Initialize:

$d = 1 \dots D$:

$max_d := r_d$;

$min_d := -r_d$;

for all points x do:

 for $d = 1 \dots D$ do:

  $y_d = x + r\,e_d$; $\bar{y}_d = x - r\,e_d$;

  $w_d = y_d - \bar{o}$; $\bar{w}_d = \bar{y}_d - \bar{o}$;

  if $\langle w_d, e_d \rangle > max_d$ then $max_d := \langle w_d, e_d \rangle$;

  if $\langle \bar{w}_d, e_d \rangle < min_d$ then $min_d := \langle \bar{w}_d, e_d \rangle$;

$\Rightarrow V = vol(B) = \prod_{d=1}^{D} (max_d - min_d)$.

For the computation of a T-value, one needs to know B's volume V – but not its geometry. • • •