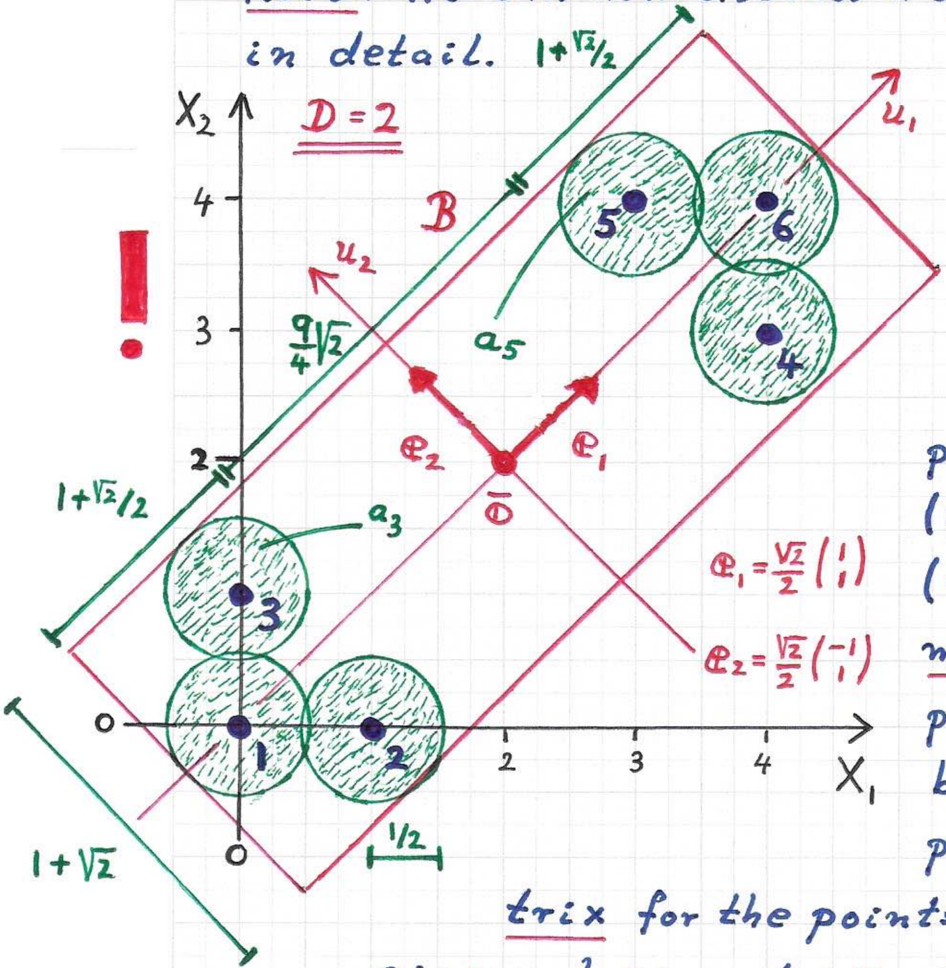
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... The described T-measure for a point set's "tightness" is the fundamental measure we use as the deciding criterion to determine whether a point set should be split into subsets. The splitting procedure should terminate when all generated subsets have a T-value above a defined threshold. We can now discuss a splitting algorithm in detail.



The left figure illustrates an example for six points '•', called 1, 2, ..., 6. The $\mathbb{X}$-space coordinates of these points are $(0,0)^T$, $(1,0)^T$, $(0,1)^T$, $(4,3)^T$, $(3,4)^T$ and $(4,4)^T$. First, we apply mean subtraction to the point set ($\overline{O} = (2,2)^T$ being the mean); compute the covariance matrix for the points; and calculate the eigenvalues and associated (normalized) eigenvectors e_1 and e_2 for the covariance matrix.

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■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

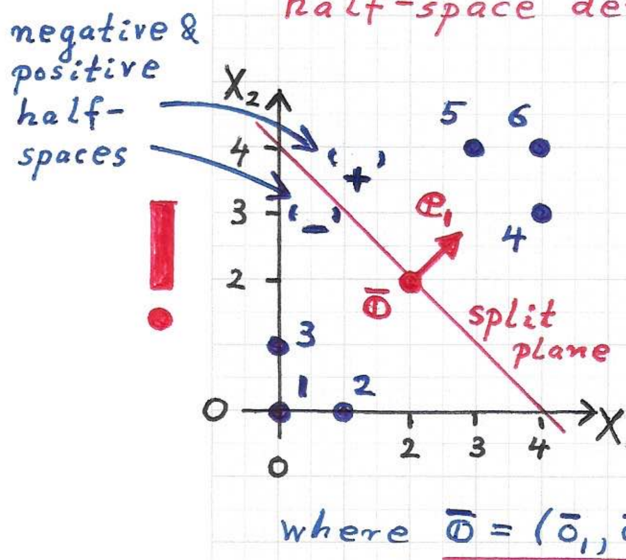
• Laplacian eigenfunctions and neural networks:... The covariance matrix of these six points, after mean subtraction, is

$$\begin{pmatrix} -2 & -1 & -2 & 2 & 1 & 2 \\ -2 & -2 & -1 & 1 & 2 & 2 \end{pmatrix}\begin{pmatrix} -2 & -1 & -2 & 2 & 1 & 2 \\ -2 & -2 & -1 & 1 & 2 & 2 \end{pmatrix}^T = \begin{pmatrix} 18 & 16 \\ 16 & 18 \end{pmatrix}.$$

Its eigenvalues are given by the solutions of the equation $(18-\lambda)^2 = 16^2$, i.e., $\lambda_1 = 34$ and $\lambda_2 = 2$. The corresponding normalized eigenvectors are $\mathbf{e}_1 = \sqrt{2}/2\,(1,1)^T$ and $\mathbf{e}_2 = \sqrt{2}/2\,(-1,1)^T$. These vectors are shown in the figure on the previous page. The figure already makes clear that the six 2-balls — each having radius $1/2$ — associated with the six points do not induce a tight packing (considering the six 2-balls inside the minimal oriented bounding box $\mathcal{B}$). The bounding box $\mathcal{B}$ can be constructed with the method described on pp. 11–15 (6/25–27/2023). The resulting box dimensions are provided in the figure; the area of box $\mathcal{B}$ is given by $V = (1+\sqrt{2}/2 + 9\sqrt{2}/4 + 1+\sqrt{2}/2)(1+\sqrt{2}) = (34+21\sqrt{2})/4$. The combined area of the six 2-balls is given by $\sum_{i=1}^{6} a_i = 6\pi(1/2)^2 = \frac{3}{2}\pi$. Thus, the T-value is $T = (\frac{3}{2}\pi)/((34+21\sqrt{2})/4) = \ldots = \frac{3}{137}\pi(34-21\sqrt{2}) \approx 0.30$. For example, if one required that $T > 1/2$, one would have to split the six-point data set into point subsets.

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... If one wants to employ a splitting approach that is binary in nature, one will be able to continue as follows: Eigenvalue λ_1 is maximal; thus, eigenvector $\mathbf{e}_1$ defines the "direction of the point set with maximal (coordinate value) variation. One should therefore use the hyper-plane that contains the point $\overline{\mathbf{o}}$ and has $\mathbf{e}_1$ as its normal vector for splitting the point set into two point subsets — where one subset will lie in the negative and one subset will lie in the positive half-space defined by the hyper-plane. The



Left figure illustrates this approach for the six-point example. The implicit definition of the hyper-plane is $f(X_1, X_2) = \langle \mathbf{e}_1, \mathbf{X} - \overline{\mathbf{o}} \rangle = 0$. In other words, every point $(X_1, X_2)^T$ in the hyper-plane satisfies the equation $e_1^1 (X_1 - \bar{o}_1) + e_2^1 (X_2 - \bar{o}_2) = 0$, where $\overline{\mathbf{o}} = (\bar{o}_1, \bar{o}_2)^T$ and $\mathbf{e}_1 = (e_1^1, e_2^1)^T$. Here, $\overline{\mathbf{o}} = (2, 2)^T$, and, for the purpose of splitting data into negative and positive half-spaces, we can use the eigenvector $\mathbf{e}_1 = (1, 1)^T$ (not normalized). We determine the sign of $f(X_1, X_2) = (X_1 - 2) + (X_2 - 2)$ for $(X_1, X_2) \in \{1, 2, 3, 4, 5, 6\}$. The value of f is negative (positive) for 1, 2 and 3 (4, 5 and 6).

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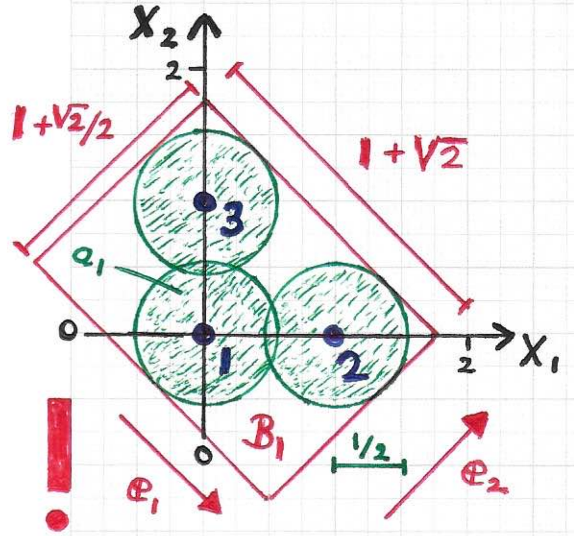
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:...



At this point, we have constructed two point subsets, i.e., $\{1,2,3\}$ and $\{3,4,5\}$. Again, we must compute the T-values for both subsets to determine whether additional splitting is necessary. The left figure illustrates the minimal oriented bounding box B_1 needed to calculate the T-value for this point subset. (The T-value computation for the second point subset, $\{4,5,6\}$ is done analogously, considering the fact that the two point subset arrangements are symmetric.) Using the dimensions of B_1, the box volume is $V_1 = (1+\sqrt{2})(1+\sqrt{2}/2) = 2+\frac{3}{2}\sqrt{2}$, and $\sum_{i=1}^{3} a_i = 3\pi(\frac{1}{2})^2 = \frac{3}{4}\pi$. Thus, the resulting T-value for this point subset is $T = (\frac{3}{4}\pi) / (2+\frac{3}{2}\sqrt{2}) = (3\pi)/(8+6\sqrt{2}) \approx 0.57$. As a consequence of symmetry, the "tightness" of point subset $\{4,5,6\}$ is $T \approx 0.57$ as well. (Box B_1 is obtained by subtracting the mean of points 1, 2 and 3, i.e., $(1/3, 1/3)^T$, from 1, 2 and 3 and performing the discussed eigenanalysis for the resulting points. This step yields eigenvalues 1 and 1/3 with corresponding normalized eigenvectors $\sqrt{2}/2\,(1,-1)^T$ and $\sqrt{2}/2\,(1,1)^T$ — defining the minimal oriented box B_1 shown in the figure above. Box B_2 is generated for $\{4,5,6\}$.)

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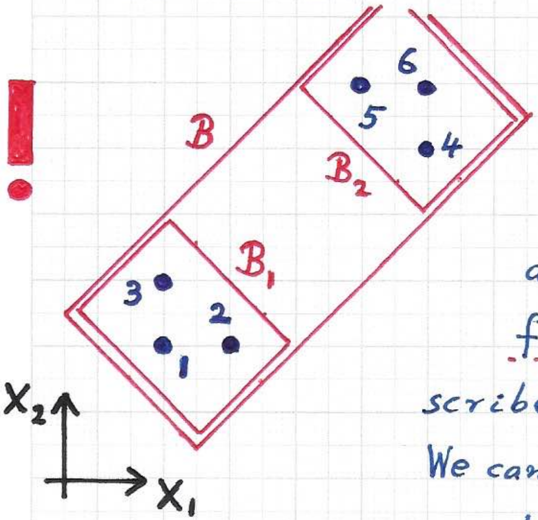
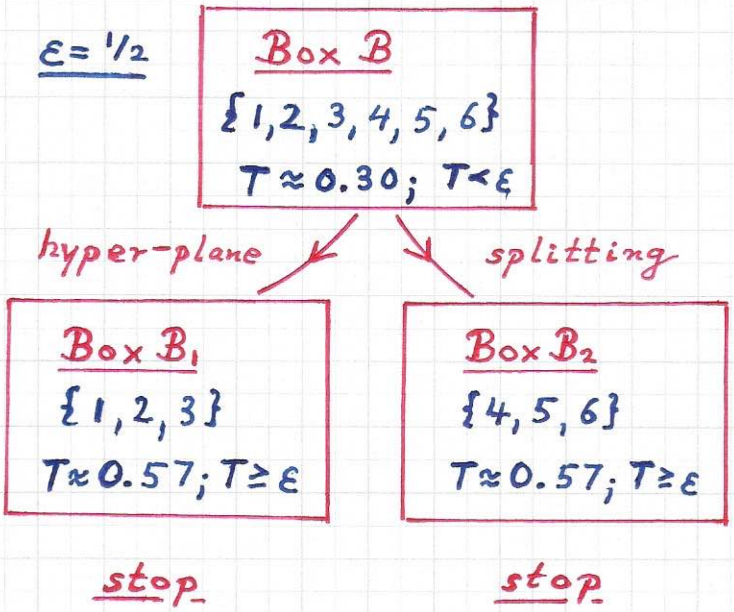
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks: ...



We can use a (binary) tree data structure to capture and represent the described hierarchical splitting process and relevant data generated as part of the process. The left figure abstractly sketches the described example involving six points. We can assume that the "tightness" requirement uses the threshold $\varepsilon = 1/2$, implying that point (sub)sets all must have a T-value above this threshold; if this requirement is not satisfied, a point (sub)set will be split. The tree structure capturing our example can be understood as a point (sub)set hierarchy shown in the left figure.

$\varepsilon = 1/2$



A box with an associated point set {•} having a T-value $T \geq \varepsilon$ is not split. The resulting point (sub)sets at the leaf level should contain more than one point.

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