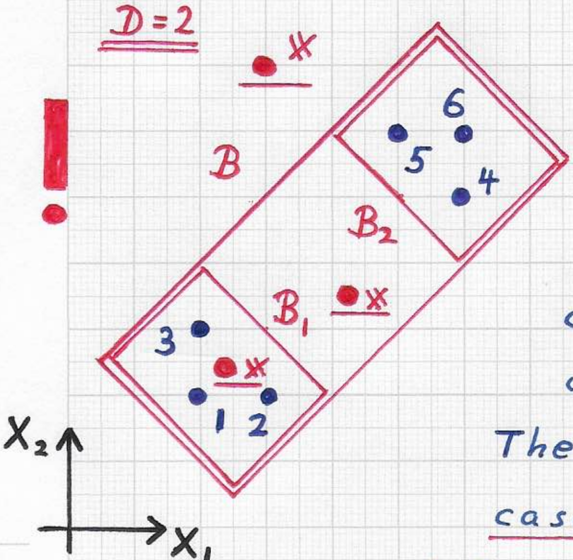
OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

- Laplacian eigenfunctions and neural networks:...

$D=2$



The constructed hierarchy of point (sub)sets and the proposed binary tree data structure for the representation of the point (sub)sets at increasingly smaller scales can directly be used for the overarching goal of classification. The left figure shows three possible cases of a (new) point ✱(●) that must be classified. The point set {1,2,3,4,5,6} must be understood as a set of "samples" that all are known to belong to the same class. Since these sample points are embedded in D-dimensional $(X_1, ..., X_D)$-space, together with the geometrical bounding box hierarchy/tree, one can use $B, B_1, B_2, ...$ to perform initial classification decisions. We consider possible cases:

i) ✱ outside B ⇒ ✱ does not belong to class.

ii) ✱ inside B AND (✱ outside B_1 AND ✱ outside B_2) ⇒ ✱ does not belong to class.

iii) ✱ inside B AND (✱ inside B_1 OR ✱ inside B_2) ⇒ ✱ potentially belongs to class.

Concerning case iii), the (new) point ✱ must be represented in normalized u-space coordinates to compute a probability value for ✱'s potential class membership.

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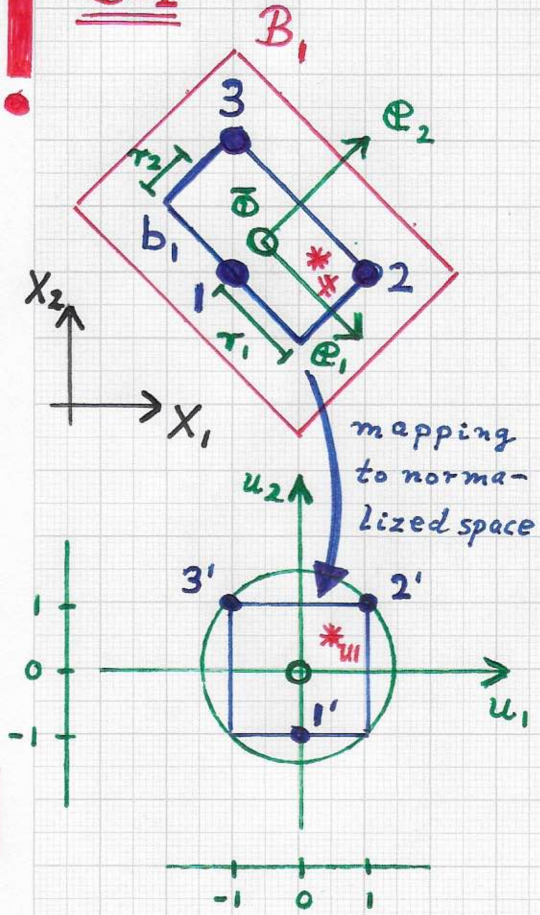
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:...

D=2



• **Note.** Considering the point (sub)set {1,2,3}, for example, we must keep in mind that two minimal bounding boxes are associated with this point (sub)set, i.e., B_1 and b_1, as shown in the left figure. The larger box, B_1, is the minimal bounding box for the three 2-balls belonging to the three points (not included in the figure). The smaller box, b_1, is the minimal bounding box for the three points. These two boxes serve two purposes: Box B_1 is used for the calculation of the "tightness" measure T for the point (sub)set; box b_1 is used for mapping the point (sub)set (and a new point to be classified!) to normalized u-space such that all points inside or on the boundary of b_1 are mapped to the interior or boundary of the hyper-cube $[-1,1]^D$ — assuming here that the to-be-classified point is inside or on the boundary of b_1 as well. The parameters for the normalizing mapping to u-space are shown in the top figure.

•••

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

- Laplacian eigenfunctions and neural networks: ... The relevant parameters and parameter values for the specific point (sub) set shown in the top figure on the previous page are the following:
  - 3 points (1,2,3): $(0,0)^T$, $(1,0)^T$, $(0,1)^T$.
  - 2 normalized eigenvectors ("eigen-directions") $\mathbb{e}_1$ and $\mathbb{e}_2$ for the point (sub) set: $\mathbb{e}_1 = \sqrt{2}/2\,(1,-1)^T$, $\mathbb{e}_2 = \sqrt{2}/2\,(1,1)^T$.
    (Note: These eigenvectors are the result of subtracting the mean, $(1/3, 1/3)^T$, from the point (sub)set and computing the eigen-values and eigenvectors for the covariance matrix.)
  - 2 minimal-oriented-box "dimension values," $r_1 = \sqrt{2}/2$, $r_2 = \sqrt{2}/4$.
  - The local origin $\overline{\mathbb{O}}$, i.e., center/average of the corner points of the point (sub)set's minimal oriented bounding box b_1: $\overline{\mathbb{O}} = (1/4, 1/4)^T$.
    (Note: It is important to keep in mind that this box b_1 center $\overline{\mathbb{O}}$ is NOT the mean of the points of the given point (sub) set.)

As described before, the parameters $\overline{\mathbb{O}}, \mathbb{e}_1, \mathbb{e}_2, r_1, r_2$ define the needed linear transformation of points from $\mathbb{X}$-space to $\mathbb{u}$-space representation:

$$u_d = \left(\langle \mathbb{x}, \mathbb{e}_d \rangle - \langle \overline{\mathbb{O}}, \mathbb{e}_d \rangle \right) / r_d \quad ,$$

where $\mathbb{x} = (x_1, \ldots, x_D)^T$, $\mathbb{u} = (u_1, \ldots, u_D)^T$ and $d = 1 \ldots D$.

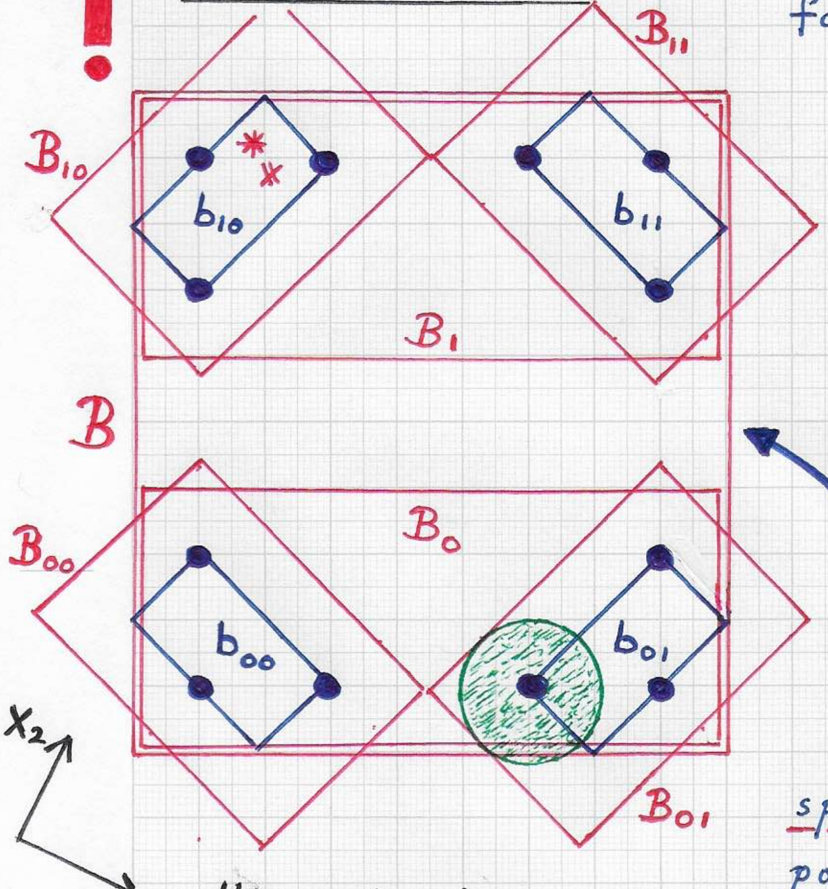
...

OBJECT AND MATERIAL EIGENFUNCTIONS - Cont'd.

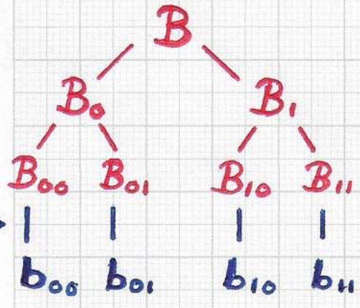
- Laplacian eigenfunctions and neural networks: ... When applying this linear transformation to the points 1, 2 and 3, one obtains their $\mathbb{u}$-space representations. The three resulting points are called 1′, 2′ and 3′ in the figure on the previous page, and their coordinate tuples are $1' = (0,-1)^T$, $2' = (1,1)^T$ and $3' = (-1,1)^T$. As discussed, a given point $\ast$ in $\mathbb{X}$-space representation is mapped to its $\mathbb{u}$-space representation $(u_1, \ldots, u_D)^T$, based on the specific linear transformation for the point (sub)set $\{1,2,3\}$; next, a probability value for class membership is computed for the to-be-classified point. The $\mathbb{u}$-tuple of the to-be-classified point determines the probability: If $\|\mathbb{u}\|^2 = u_1^2 + \ldots + u_D^2 < D$, then a probability value greater than zero will be assigned; if $\|\mathbb{u}\|^2 \geq D$, then zero will be assigned to the point — keeping in mind that this zero-probability assignment is only relative to the point (sub)set $\{1,2,3\}$; of course, one must consider all other point (sub)sets and their corresponding mappings / linear transformations to normalized $\mathbb{u}$-space to determine the final value of probability for the to-be-classified point's class membership. The figure on the previous page refers to the to-be-classified point as '$\ast$', $\ast$ and $\mathbb{u}$. ...

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

- Laplacian eigenfunctions and neural networks: ...



Hierarchy of minimal oriented bounding boxes.



Binary tree used to represent hierarchy of all bounding boxes.

We consider a simple example for $D = 2$, to summarize the main concepts used for the construction of the described hierarchical approach for organizing a point set {•} via a set of minimal oriented bounding hyper-boxes. Here, 12 points (representing "samples," "instances" of a specific class) are given. Repeated splitting is applied to the point (sub)set(s), until the "tightness" values T all satisfy a threshold condition. The (red) bounding boxes B, B_0, B_1, B_{01}, B_{01}, B_{10} and B_{11} are bounding boxes of the 2-balls associated with certain point subsets. The left figure also shows the resulting hierarchical binary tree representation. Eventually, at the leaf level, the point subset bounding boxes are represented. The point ⁎ must be classified.

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