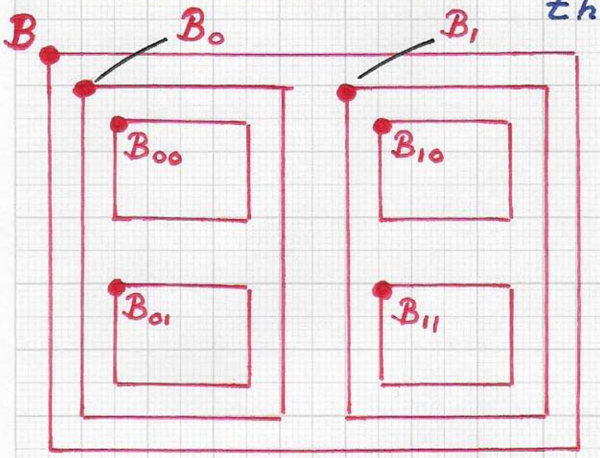
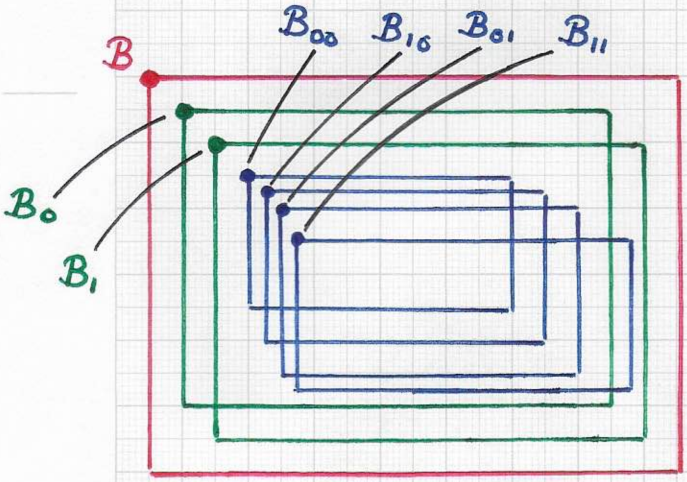
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

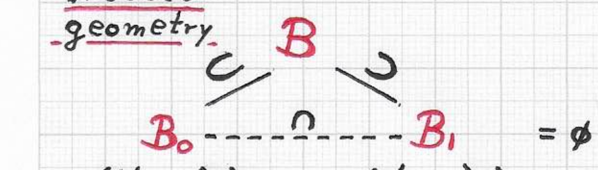
• Laplacian eigenfunctions and neural networks: ...





• **Note.** The left figure serves the purpose of explaining a specific aspect of the desribed classification decision-making process that relates to the geometry of the bounding boxes B; B_0, B_1; and B_{00}, B_{01}, B_{10}, B_{11}. The top portion of the figure illustrates the "ideal" geomtrical case: the hierarchical tree (and indexing) organization of the boxes is also geometrically reflected by a hierarchical nesting of spatial regions. In the following, a box is understood as the set of (interior) points in space bounded by the box. Thus, the arrangement of boxes in the top portion satisfies:

Nested geometry

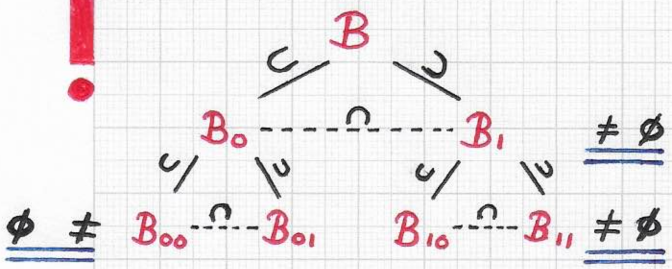


$$\boxed{\begin{array}{l} B \supset B_0 \ \wedge\ B \supset B_1 \ \wedge\ B_0 \cap B_1 = \emptyset \\ \wedge\ B_0 \supset B_{00} \ \wedge\ B_0 \supset B_{01} \ \wedge\ B_{00} \cap B_{01} = \emptyset \\ \wedge\ B_1 \supset B_{10} \ \wedge\ B_1 \supset B_{11} \ \wedge\ B_{10} \cap B_{11} = \emptyset \end{array}}$$

The left figure is a tree-based representation of this "perfect nesting." ...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• **Laplacian eigenfunctions and neural networks:...** One should assume that such a "perfect nesting" is often satisfied by D-dimensional point data sets that are repeatedly split into subsets via the described methods for constructing minimal oriented bounding boxes and subdividing a point (sub)set into two point subsets. Nevertheless, it is possible that the "perfect nesting" condition — as expressed on the previous page via set theory and Boolean logic, for a specific 2-dimensional scenario — is not satisfied. Whenever one or more than one of the terms in the Boolean AND ('∧') expression on the previous page are FALSE, the expression is FALSE, and the particular given tree hierarchy of bounding boxes does not represent a "perfect nesting." The bottom portion of the figure on the previous page is a sketch of a geometrical box arrangement scenario where



Tree representation of Boolean expression (right).

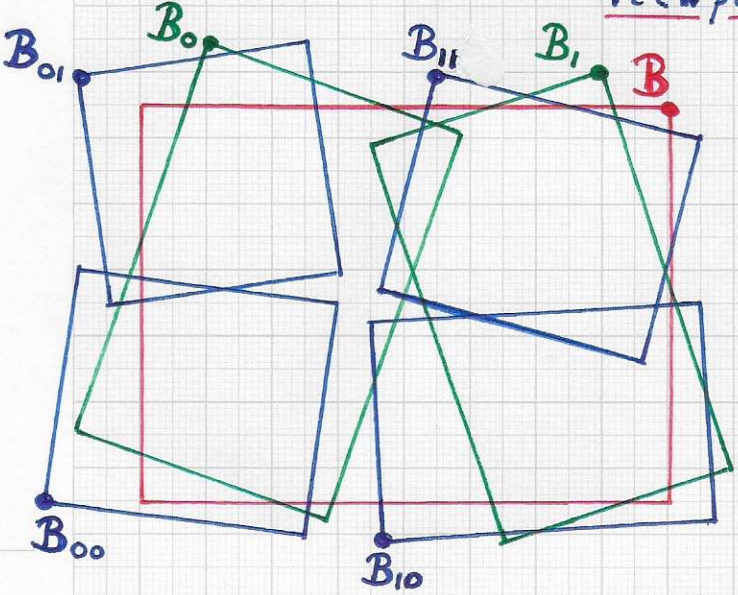
$$\boxed{\begin{array}{l} B \supset B_0 \wedge B \supset B_1 \wedge \underline{B_0 \cap B_1 \neq \emptyset} \\ \wedge B_0 \supset B_{00} \wedge B_0 \supset B_{01} \wedge \underline{B_{00} \cap B_{01} \neq \emptyset} \\ \wedge B_1 \supset B_{10} \wedge B_1 \supset B_{11} \wedge \underline{B_{10} \cap B_{11} \neq \emptyset} \end{array}}$$

The left figure represents this arrangement and implied Boolean expression.

• • •

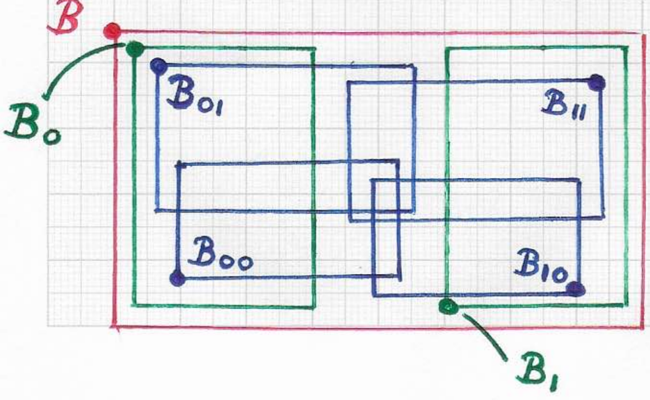
OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

- Laplacian eigenfunctions and neural networks: ...



From a purely combinatorial viewpoint, one must consider all combinatorially possible relationships between pairs of bounding boxes, as far as the relationships '$\supset$' (superset) and '$\cap$' (intersection) are concerned. The left figure illustrates the same box tree hierarchy as used on the previous two pages — as far as the tree data structure is concerned — but extremely different geometrical '$\supset$' and '$\cap$' characteristics. **Here,** the following is **TRUE:**

$$\boxed{\begin{array}{l} \ B \not\supset B_0 \ \wedge\ B \not\supset B_1 \ \wedge\ B_0 \cap B_1 \neq \emptyset \\ \wedge\ B_0 \not\supset B_{00} \ \wedge\ B_0 \not\supset B_{01} \ \wedge\ B_{00} \cap B_{01} \neq \emptyset \\ \wedge\ B_1 \not\supset B_{10} \ \wedge\ B_1 \not\supset B_{11} \ \wedge\ B_{10} \cap B_{11} \neq \emptyset \end{array}} \ .$$



Another interesting case is shown in the left figure, where, for example, the following relationships are **TRUE:**

$$\boxed{\begin{array}{l} B \supset B_0 \wedge B \supset B_1 \wedge B_0 \cap B_1 = \emptyset \\ \wedge B_0 \not\supset B_{00} \wedge B_0 \not\supset B_{01} \\ \wedge B_1 \not\supset B_{10} \wedge B_1 \not\supset B_{11} \end{array}} \ \wedge \ \ldots$$

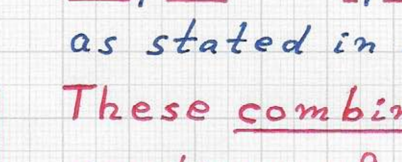
OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

- Laplacian eigenfunctions and neural networks:... [continuation of discussion of case from previous page]

$$\boxed{\begin{array}{l} \wedge\ B_{00} \cap B_{01} \neq \emptyset \ \wedge\ B_{00} \cap B_{10} \neq \emptyset \ \wedge\ B_{00} \cap B_{11} \neq \emptyset \\ \wedge\ B_{10} \cap B_{01} \neq \emptyset \ \wedge\ B_{10} \cap B_{11} \neq \emptyset \ \wedge\ B_{01} \cap B_{11} \neq \emptyset \\ \wedge\ \bigcap(B_{00}, B_{01}, B_{10}) \neq \emptyset \ \wedge\ \bigcap(B_{00}, B_{01}, B_{11}) \neq \emptyset \\ \wedge\ \bigcap(B_{00}, B_{10}, B_{11}) \neq \emptyset \ \wedge\ \bigcap(B_{01}, B_{10}, B_{11}) \neq \emptyset \\ \wedge\ \bigcap(B_{00}, B_{01}, B_{10}, B_{11}) \neq \emptyset \end{array}}$$

For example, one interesting fact about this case is the "mutual intersection of all possible pairs, triples and quadruples of B_{ij} spatial regions," as stated in the Boolean expression above.

These combinatorially possible (non-empty) intersections of bounding boxes B, B_0, B_1, B_{00}, B_{01}, B_{10} and B_{11} must be considered (as well as the possible superset and subset relationships) as the linear transformations (translation, rotation, scaling) defined by eigenanalyses of point set can indeed create the "non-perfect nesting cases" described via the above geometrical examples. When the repeated splitting process of point (sub)sets does not produce a "perfect nesting" of boxes, see left figure, the classification of * becomes more complex. ...

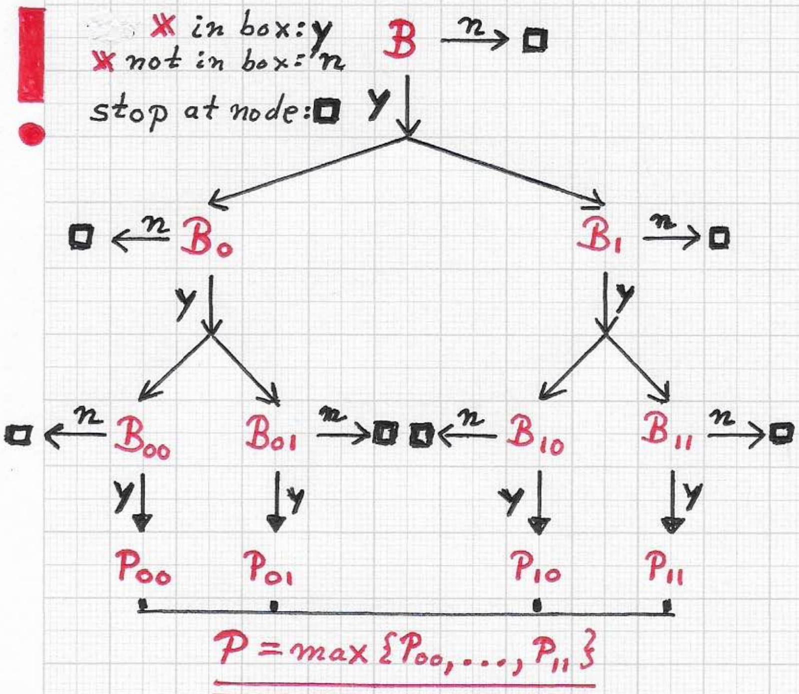


Three scenarios of location of * (to be classified).

$$B_0 \subset B \supset B_1, \quad B_0 \cap B_1 \neq \emptyset$$

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

- Laplacian eigenfunctions and neural networks:... The simple example illustrated at the bottom of the previous page, involving only a single point set splitting step, shows the case where a point $*$, with $* \in \mathcal{B}$, when lying inside a sub-box of $\mathcal{B}$, could satisfy one of the following three sub-box inclusion conditions: (i) $* \in \mathcal{B}_0 \wedge * \notin \mathcal{B}_1$; (ii) $* \notin \mathcal{B}_0 \wedge * \in \mathcal{B}_1$; (iii) $* \in \mathcal{B}_0 \wedge * \in \mathcal{B}_1$. When having to classify a point $*$ that satisfies (iii), one must determine two class membership probabilities for $*$ — one based on the $\mathcal{B}_0$ node and one based on the $\mathcal{B}_1$ node in the splitting tree.



As a consequence of such geometrically possible box arrangements, one must be prepared to traverse multiple of the top-down paths in the box tree, starting at $\mathcal{B}$, see left figure. For example, it is possible that four non-zero values for $P_{00}, P_{01}, P_{10}, P_{11}$ result.

Further, from a combinatorial viewpoint, $\mathcal{B}_{00}$ could lie outside $\mathcal{B}_0$ and $\mathcal{B}_0$ could lie outside $\mathcal{B}$ — but those arrangements are geometrically not viable. ...