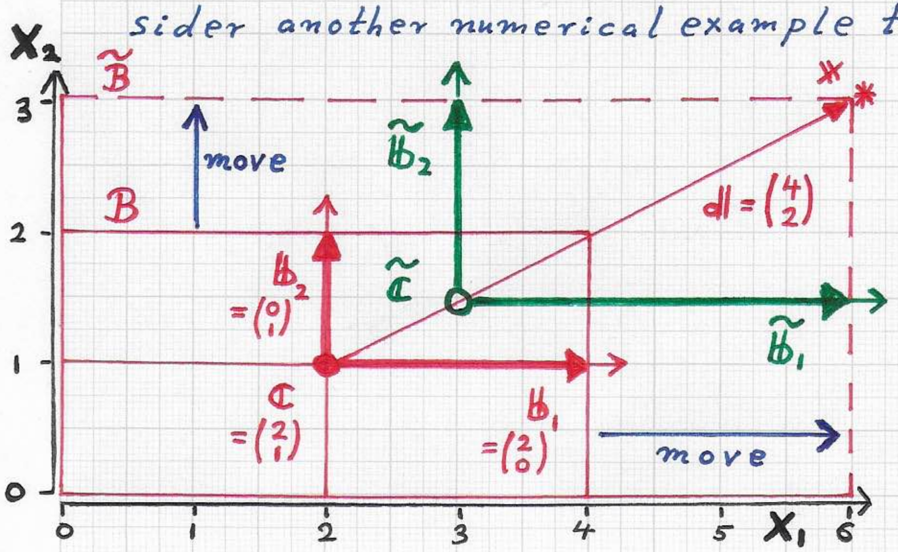
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

- Laplacian eigenfunctions and neural networks:... Generally, one should not scale the box B "more than necessary," i.e., one should move outward only one or two of B's boundary edges (in the 2D box example considered here); the outward movement of one or two edges must be done via a box scaling that ensures that the point $*$ will lie on the boundary of the scaled box. As a consequence of the scaling operation, one will also have to properly update the location of the box center and one or multiple box-inherent basis vectors. We consider another numerical example to explain these steps.



The left figure shows the original box B and the box-inherent system defined by $\{c, b_1, b_2\}$. The considered point $*$ has global coordinates $* = (X_1, X_2)^T = (6, 3)^T$. When calculating the point's **local** box coordinates one obtains the tuple $*_{loc} = (2, 2)^T$, i.e., $d = (4, 2)^T = 2\, b_1 + 2\, b_2 = \alpha_1 b_1 + \alpha_2 b_2$. Since $|\alpha_1| > 1$ and $|\alpha_2| > 1$, two edges of B must be moved outward.

...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

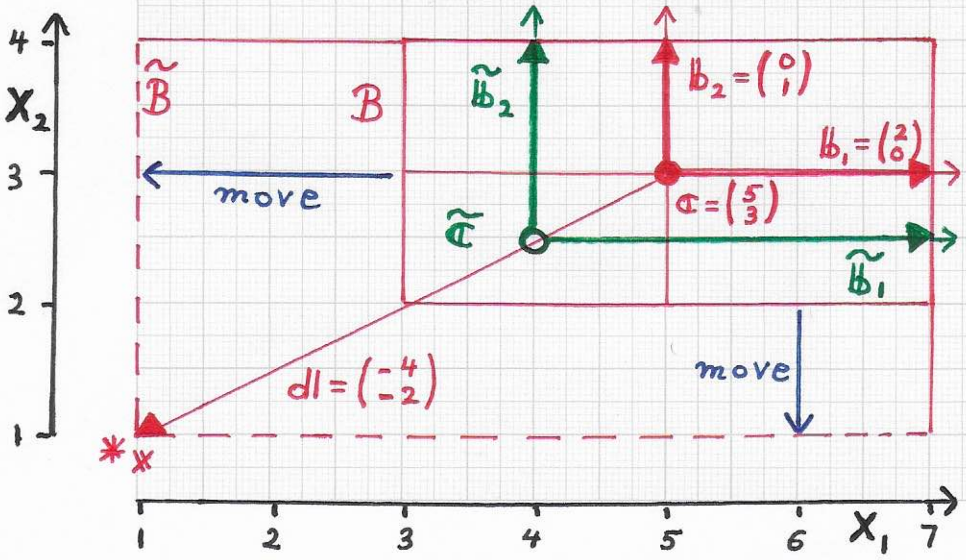
• Laplacian eigenfunctions and neural networks:... The figure on the previous page refers to the new box coordinate system as $\{\tilde{\mathbb{c}}, \tilde{\mathbb{b}}_1, \tilde{\mathbb{b}}_2\}$. The values of α_1 and α_2 determine the values needed for the updated box coordinate system. The relatively simple geometrical conditions to be satisfied by the scaled box, resulting from moving the right and top edges of $\mathbb{B}$, define the following equations for $\tilde{\mathbb{c}}$, $\tilde{\mathbb{b}}_1$ and $\tilde{\mathbb{b}}_2$:

$$\tilde{\mathbb{c}} = \mathbb{c} + \frac{\alpha_1 - 1}{2}\mathbb{b}_1 + \frac{\alpha_2 - 1}{2}\mathbb{b}_2 = \begin{pmatrix}2\\1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}2\\0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}3\\1.5\end{pmatrix};$$

$$\tilde{\mathbb{b}}_1 = \frac{\alpha_1 + 1}{2}\mathbb{b}_1 = 3/2\,\mathbb{b}_1 = 3/2\begin{pmatrix}2\\0\end{pmatrix} = \begin{pmatrix}3\\0\end{pmatrix};$$

$$\tilde{\mathbb{b}}_2 = \frac{\alpha_2 + 1}{2}\mathbb{b}_2 = 3/2\,\mathbb{b}_2 = 3/2\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1.5\end{pmatrix}.$$

Consequently, one obtains $\tilde{\mathbb{d}} = (6,3)^T - (3,1.5)^T = (3, 1.5)^T = 1 \cdot \tilde{\mathbb{b}}_1 + 1 \cdot \tilde{\mathbb{b}}_2$ as expected. The left figure shows another example to emphasize that one must use signs properly when calculating the new system. Depending on the edge(s) to be moved, one must use signs accordingly. ...



Moving left and bottom edges of $\mathbb{B}$.

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

- Laplacian eigenfunctions and neural networks: … In this example, the point $\mathbf{x}$ has the global coordinates $\mathbf{x} = (X_1, X_2)^T = (1,1)^T$ and also lies outside box $\mathcal{B}$. The relevant positional vector to be expressed with respect to the box basis vectors is $\mathbf{d} = (-4,-2)^T$. Since $\mathbf{d}$ can be written as $\mathbf{d} = -2\mathbf{b}_1 - 2\mathbf{b} = \alpha_1 \mathbf{b}_1 + \alpha_2 \mathbf{b}_2$, the point's local box representation is $\mathbf{x}_{Loc} = (-2,-2)^T$. Again, $|\alpha_1| > 1$ and $|\alpha_2| > 1$, and one must move two of the edges of $\mathcal{B}$. As the values of both α_1 and α_2 are negative, one must move the left and bottom edges of $\mathcal{B}$ – such that the point will become the location of the lower-left vertex of the scaled box $\tilde{\mathcal{B}}$. In this case, one must correctly use signs in the calculations of the values for $\{\tilde{\mathbf{c}}, \tilde{\mathbf{b}}_1, \tilde{\mathbf{b}}_2\}$. The slightly "adjusted" equations for the new system are:

$$\tilde{\mathbf{c}} = \mathbf{c} + \frac{\alpha_1+1}{2}\mathbf{b}_1 + \frac{\alpha_2+1}{2}\mathbf{b}_2 = \begin{pmatrix}5\\3\end{pmatrix} - \frac{1}{2}\begin{pmatrix}2\\0\end{pmatrix} - \frac{1}{2}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}4\\2.5\end{pmatrix};$$

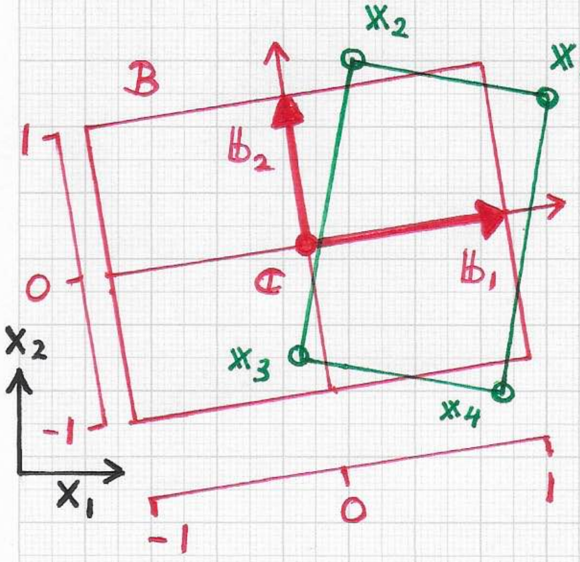
$$\tilde{\mathbf{b}}_1 = \frac{-\alpha_1+1}{2}\mathbf{b}_1 = \frac{3}{2}\begin{pmatrix}2\\0\end{pmatrix} = \begin{pmatrix}3\\0\end{pmatrix};$$

$$\tilde{\mathbf{b}}_2 = \frac{-\alpha_2+1}{2}\mathbf{b}_2 = \frac{3}{2}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1.5\end{pmatrix}.$$

These elements of the local coordinate system of box $\tilde{\mathcal{B}}$ are also shown in the figure on the previous page. Relatively to this system, one obtains the point's positional vector as $\tilde{\mathbf{d}} = (1,1)^T - (4,2.5)^T = (-3,-1.5)^T = -1\cdot\tilde{\mathbf{b}}_1 - 1\cdot\tilde{\mathbf{b}}_2$. Thus, the point's new coordinates are $\tilde{\mathbf{x}}_{Loc} = (-1,-1)^T$ as expected. …

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... In fact, it is even simpler to understand the necessary expansion of the box B as the consequence of performing the splitting of B into two child boxes, while ensuring that all corner vertices of these child boxes lie inside B — or a minimally expanded version of B, called $\tilde{B}$. This viewpoint can be explained in some detail via the left figure.



Here, a child box of box B has vertices $\mathbb{x}_1, \mathbb{x}_2, \mathbb{x}_3$ and $\mathbb{x}_4$. Several of these vertices lie outside B. One can represent these four vertices with respect to the box coordinate system $\{\mathbb{c}, \mathbb{b}_1, \mathbb{b}_2\}$, i.e.,

$$\mathbb{x}_i = \mathbb{c} + \alpha_1^i \mathbb{b}_1 + \alpha_2^i \mathbb{b}_2, \; i = 1 \ldots 4.$$

Thus, the 2D coordinate tuple of a vertex is $(\alpha_1^i, \alpha_2^i)^T$. A point lies inside (or on the boundary of) B when the values of all its box-system coordinates lie in the interval $[-1, 1]$, indicated in the figure. We consider the minimal and maximal values of the coords. of points $\mathbb{x}_i$ to determine whether box B must be expanded and by what amount. Therefore, we calculate

$$\alpha_1^{MIN} = \min\{\alpha_1^i\}; \; \alpha_1^{MAX} = \max\{\alpha_1^i\}; \; \alpha_2^{MIN} = \min\{\alpha_2^i\}; \; \alpha_2^{MAX} = \max\{\alpha_2^i\},$$
$$i = 1 \ldots 4.$$

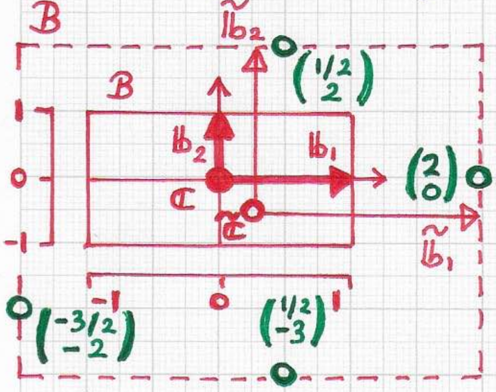
...

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... The initial extremal α-values of the box $\mathbb{B}$, i.e., the four box corner vertices, are $\alpha_1^{min} = -1$; $\alpha_1^{max} = 1$; $\alpha_2^{min} = -1$; $\alpha_2^{max} = 1$. One must update these values – potentially – based on the box-system coordinate value extrema of $\mathbf{x}_i$ using the following conditions:

$$\text{IF } \alpha_1^{MIN} < \alpha_1^{min} \text{ THEN } \alpha_1^{min} := \alpha_1^{MIN};$$
$$\text{IF } \alpha_1^{MAX} > \alpha_1^{max} \text{ THEN } \alpha_1^{max} := \alpha_1^{MAX};$$
$$\text{IF } \alpha_2^{MIN} < \alpha_2^{min} \text{ THEN } \alpha_2^{min} := \alpha_2^{MIN};$$
$$\text{IF } \alpha_2^{MAX} > \alpha_2^{max} \text{ THEN } \alpha_2^{max} := \alpha_2^{MAX}.$$

If at least one of these potential extremal value updates is performed, one will have to modify the box coordinate system to achieve the needed expansion. First, one computes $\alpha_1^{mid} = (\alpha_1^{min} + \alpha_1^{max})/2$ and $\alpha_2^{mid} = (\alpha_2^{min} + \alpha_2^{max})/2$.



Box and expanded box.

These "mid-values" define the center $\tilde{\mathbb{C}}$ for the new box $\tilde{\mathbb{B}}$ as $\tilde{\mathbb{C}} = (\alpha_1^{mid}, \alpha_2^{mid})^T$. Second, one calculates scaling factors s_1 and s_2, i.e., $s_1 = \alpha_1^{max} - \alpha_1^{mid}$ and $s_2 = \alpha_2^{max} - \alpha_2^{mid}$. Thus, the box basis vectors of $\tilde{\mathbb{B}}$ are $\tilde{\mathbb{b}}_1 = s_1 \mathbb{b}_1$ and $\tilde{\mathbb{b}}_2 = s_2 \mathbb{b}_2$. The left figure illustrates a simple 4-point example. Here, $\alpha_1^{MIN} = -3/2$, $\alpha_1^{MAX} = 2$, $\alpha_2^{MIN} = -3$ and $\alpha_2^{MAX} = 2$. The resulting new system for $\tilde{\mathbb{B}}$ is given by $\tilde{\mathbb{C}} = \frac{1}{2}(-3/2+2, -3+2)^T = (1/4, -1/2)^T$, $\tilde{\mathbb{b}}_1 = 7/4\, \mathbb{b}_1$ and $\tilde{\mathbb{b}}_2 = 5/2\, \mathbb{b}_2$. These box expansions can lead to box-box intersections. ...