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8/19/2023

## OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

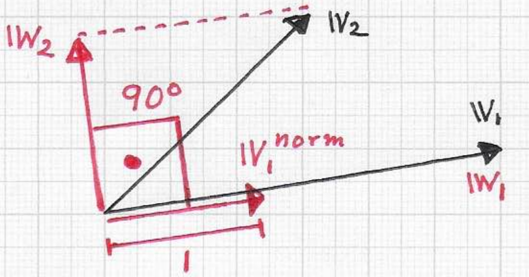
- **Laplacian eigenfunctions and neural networks:...** The orthoNORMALization procedure applied to $D$ linearly independent vectors $\mathbf{v}_1, \dots, \mathbf{v}_D$ performs these steps:

- $\mathbf{w}_1 := \mathbf{v}_1$ ; $\mathbf{w}_1 := \mathbf{w}_1 / \|\mathbf{w}_1\|$ ;
- $\mathbf{w}_2 := \mathbf{v}_2 - \langle \mathbf{v}_2, \mathbf{w}_1 \rangle \mathbf{w}_1$ ; $\mathbf{w}_2 := \mathbf{w}_2 / \|\mathbf{w}_2\|$ ;
- $\mathbf{w}_3 := \mathbf{v}_3 - \langle \mathbf{v}_3, \mathbf{w}_1 \rangle \mathbf{w}_1 - \langle \mathbf{v}_3, \mathbf{w}_2 \rangle \mathbf{w}_2$ ; $\mathbf{w}_3 := \mathbf{w}_3 / \|\mathbf{w}_3\|$ ;

...

- $\mathbf{w}_D := \mathbf{v}_D - \sum_{d=1}^{D-1} \langle \mathbf{v}_D, \mathbf{w}_d \rangle \mathbf{w}_d$ ; $\mathbf{w}_D := \mathbf{w}_D / \|\mathbf{w}_D\|$ .

For our purposes, we only need to perform orthoGONALization; a NORMALization of the resulting vectors is not desired. "Orthogonalization-only" is illustrated for the 2D case in the left figure. Here, one performs the steps




- $\mathbf{w}_1 := \mathbf{v}_1$ ; $\mathbf{w}_1^{norm} := \mathbf{w}_1 / \|\mathbf{w}_1\|$;
- $\mathbf{w}_2 := \mathbf{v}_2 - \langle \mathbf{v}_2, \mathbf{w}_1^{norm} \rangle \mathbf{w}_1^{norm}$ ; $\{ \mathbf{w}_2^{norm} := \mathbf{w}_2 / \|\mathbf{w}_2\|; \}$.

Therefore, the "orthoGONALIZATION-ONLY" procedure applied to $D$ linearly independent vectors $\mathbf{v}_1, \dots, \mathbf{v}_D$ performs the following steps:

- $\mathbf{w}_1 := \mathbf{v}_1$ ; $\mathbf{w}_d := \mathbf{v}_d - \sum_{i=1}^{d-1} \langle \mathbf{v}_d, \mathbf{w}_i^{norm} \rangle \mathbf{w}_i^{norm}$ , $d = 2 \dots D$,

  where $\mathbf{w}_i^{norm} := \mathbf{w}_i / \|\mathbf{w}_i\|$ , $i = 1 \dots (D-1)$.



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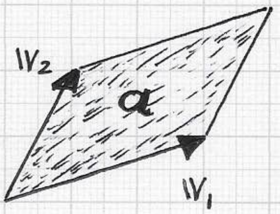
## ▪ OBJECT AND MATERIAL EIGENFUNCTIONS - Cont'd.

• Laplacian eigenfunctions and neural networks: ...

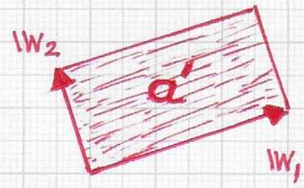



$\mathbf{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\underline{a = 5}$

$\mathbf{w}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$




$\mathbf{w}_2 = \begin{pmatrix} -1/2 \\ 3/2 \end{pmatrix}$

$\underline{a' = 5}$

In our context, it can be algorithmically enforced that the vectors $\mathbf{v}_1, \dots, \mathbf{v}_D$ are mutually "nearly orthogonal"; but we still must perform necessary orthonormalization (to define a needed orthogonal basis vector set for a local bounding box). Thus, one might want to ensure that an orthogonalization procedure preserves certain length or (hyper-) volume properties. The left figure sketches an example for the 2D setting. When applying the described orthogonalization procedure to the vector $\mathbf{v}_1 = (3,1)^T$ and $\mathbf{v}_2 = (1,2)^T$, one obtains $\mathbf{w}_1 := \mathbf{v}_1 = (3,1)^T$ and $\mathbf{w}_2 := \mathbf{v}_2 - \langle \mathbf{v}_2, \mathbf{w}_1^{norm} \rangle \mathbf{w}_1^{norm}$
$= (1,2)^T - \langle (1,2)^T, (3,1)^T/\sqrt{10} \rangle \cdot (3,1)^T/\sqrt{10} = (1,2)^T - \frac{1}{2}(3,1)^T$
$= (1,2)^T - (3/2, 1/2)^T = (-1/2, 3/2)^T$.

The vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ define the parallelepiped shown in the above figure, defining the area $a$ (shaded area); one sees via area computation that $a = 5$. After orthogonalization, the vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ also define a parallelepiped (rectangle), defining the area $a'$ (shaded area); one obtains $a' = 5 = a$.

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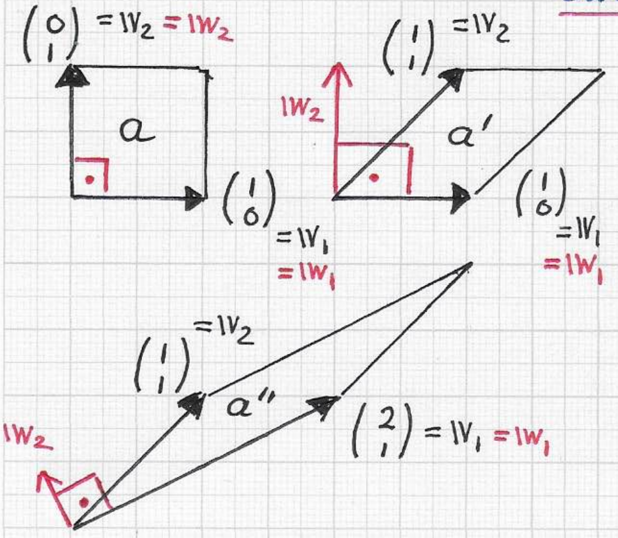
## ■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:...




Three parallelepipeds in 2D space, where $a = a' = a''$.

• **Note.** The (positive) hyper-volume of a parallelepiped region in $D$-dimensional space with linearly independent "parallelepiped basis vectors" $\mathbf{v}_1, \ldots, \mathbf{v}_D$ is given by the absolute value of the determinant of the matrix $P$ having $\mathbf{v}_1, \ldots, \mathbf{v}_D$ as its columns:

$$v = vol(\mathbf{v}_1, \ldots, \mathbf{v}_D) = |\det P|,$$

where $P = \begin{bmatrix} \mathbf{v}_1 & \ldots & \mathbf{v}_D \end{bmatrix}$. The top-left figure shows three examples of parallelepipeds in the plane. The areas $a$, $a'$ and $a''$ are

$$a = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1, \quad a' = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \quad \text{and} \quad a'' = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 1.$$

When applying orthogonalization to the pair of vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in these three examples, one obtains the three vector pairs $\mathbf{w}_1 = (1,0)^T$, $\mathbf{w}_2 = (0,1)^T$ in the first example; $\mathbf{w}_1 = (1,0)^T$, $\mathbf{w}_2 = (0,1)^T$ in the second example; and $\mathbf{w}_1 = (2,1)^T$, $\mathbf{w}_2 = (-1/5, 2/5)^T$ in the third example. The vector pairs $\mathbf{w}_1$ and $\mathbf{w}_2$ define rectangles, and they all have and thus preserve the original parallelepiped area, i.e., **1**.

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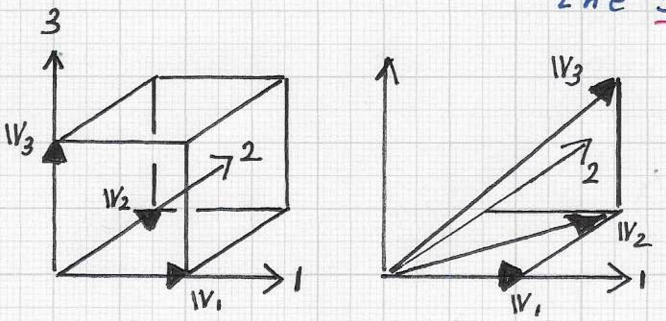
## OBJECT AND MATERIAL EIGENFUNCTIONS - Cont'd.

- Laplacian eigenfunctions and neural networks: ...

We consider a simple example for the 3D case, where two parallelepipeds have volume 1. The left figure shows the three vectors $\mathbb{v}_i$, $i=1,2,3$, defining the two parallelepipeds. After orthogonalization, one obtains the new orthogonal vectors $\mathbb{w}_i$, $i=1,2,3$, and is evident that the volumes for the second configuration of the vectors $\mathbb{v}_i$ before and after orthogonalization are both 1, since



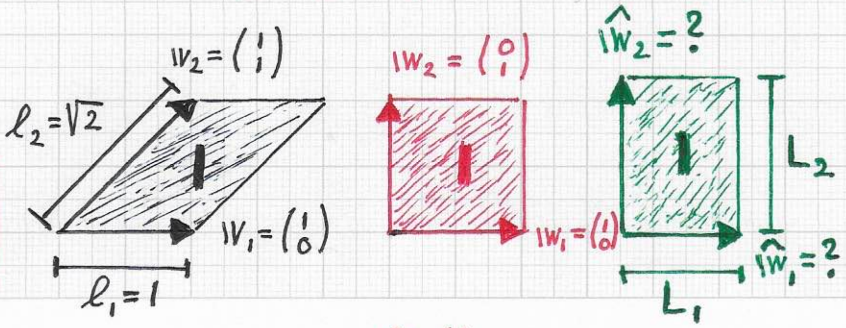

$\mathbb{v}_1 = (1,0,0)^T = \mathbb{w}_1$ $\quad$ $\mathbb{v}_1 = (1,0,0)^T = \mathbb{w}_1$

$\mathbb{v}_2 = (0,1,0)^T = \mathbb{w}_2$ $\quad$ $\mathbb{v}_2 = (1,1,0)^T$, $\mathbb{w}_2 = (0,1,0)^T$

$\mathbb{v}_3 = (0,0,1)^T = \mathbb{w}_3$ $\quad$ $\mathbb{v}_3 = (1,1,1)^T$, $\mathbb{w}_3 = (0,0,1)^T$

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1.$$

Thus, the orthogonalization procedure as described seems to preserve the (hyper-) volume of a general parallelepiped given initially by the vectors $\mathbb{v}_1, \ldots, \mathbb{v}_D$. Of course, a formal proof is required for this observed property of the orthogonalization procedure.




Parallelepiped $\quad$ Orthogonalization $\quad$ Scaling

The left figure illustrates another potentially desirable property: preservation of length ratios of vectors: $\dfrac{\ell_2}{\ell_1} = \dfrac{L_2}{L_1}$.

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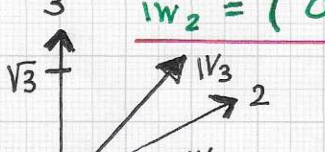
## OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

- Laplacian eigenfunctions and neural networks: … In this context, length preservation of basis vectors is understood as follows: Given basis vectors $\mathbb{v}_1$ and $\mathbb{v}_2$ with lengths $\ell_1 = \|\mathbb{v}_1\|$ and $\ell_2 = \|\mathbb{v}_2\|$, generate orthogonal vectors $\mathbb{w}_1$ and $\mathbb{w}_2$, and subsequently scale vectors $\mathbb{w}_1$ and $\mathbb{w}_2$ such that the ratio $\ell_2/\ell_1$ is preserved, i.e., calculate vectors $\hat{\mathbb{w}}_1$ and $\hat{\mathbb{w}}_2$ with lengths $L_1 = \|\hat{\mathbb{w}}_1\|$ and $L_2 = \|\hat{\mathbb{w}}_2\|$ where $L_2 / L_1 = \ell_2 / \ell_1$. The parallelepipeds defined by $\mathbb{v}_1$ and $\mathbb{v}_2$, by $\mathbb{w}_1$ and $\mathbb{w}_2$, and by $\hat{\mathbb{w}}_1$ and $\hat{\mathbb{w}}_2$ must have the same area / hyper-volume. (This concept of length preservation must be generalized to the D-dimensional case.) We now consider the specific values of the example shown in the bottom-left figure on the previous page:
  - $\mathbb{v}_1 = (1,0)^T$, $\mathbb{v}_2 = (1,1)^T$ ; $\ell_1 = \|\mathbb{v}_1\| = 1$, $\ell_2 = \|\mathbb{v}_2\| = \sqrt{2}$
    $\Rightarrow$ area $a = 1$ ; $\ell_2 / \ell_1 = \sqrt{2}$ .
  - $L_2 / L_1 = \sqrt{2} \Rightarrow L_2 = L_1\sqrt{2}$ ;
    $L_1 L_2 = a = 1 \Rightarrow L_1^2\sqrt{2} = 1 \Rightarrow L_1 = \sqrt{\sqrt{2}/2}$
    $\hat{\mathbb{w}}_1 = (L_1, 0)^T = (\sqrt{\sqrt{2}/2}, 0)^T = (0.89, 0)^T$
    $\hat{\mathbb{w}}_2 = (0, L_2)^T = (0, L_1\sqrt{2})^T = (0, \sqrt[4]{2})^T = (0, 1.19)^T$




$\ell_1 = 1$, $\ell_2 = 2$, $\ell_3 = 3$

$\Rightarrow \ell_3 / \ell_2 / \ell_1 = 3/2/1$

Another example is shown in the left figure:

- $\mathbb{v}_1 = (1,0,0)^T$, $\mathbb{v}_2 = (\sqrt{2},\sqrt{2},0)^T$, $\mathbb{v}_3 = (\sqrt{3},\sqrt{3},\sqrt{3})^T$

$$\begin{vmatrix} 1 & \sqrt{2} & \sqrt{3} \\ 0 & \sqrt{2} & \sqrt{3} \\ 0 & 0 & \sqrt{3} \end{vmatrix} = \sqrt{6}$$

- $\mathbb{w}_1 = (1,0,0)^T$, $\mathbb{w}_2 = (0,\sqrt{2},0)^T$, $\mathbb{w}_3 = (0,0,\sqrt{3})^T$
- $\hat{\mathbb{w}}_1 = (L_1,0,0)^T$, $\hat{\mathbb{w}}_2 = (0,L_2,0)^T$, $\hat{\mathbb{w}}_3 = (0,0,L_3)^T$

$\Rightarrow$ Condition: $L_3 / L_2 / L_1 = \ell_3 / \ell_2 / \ell_1$ …