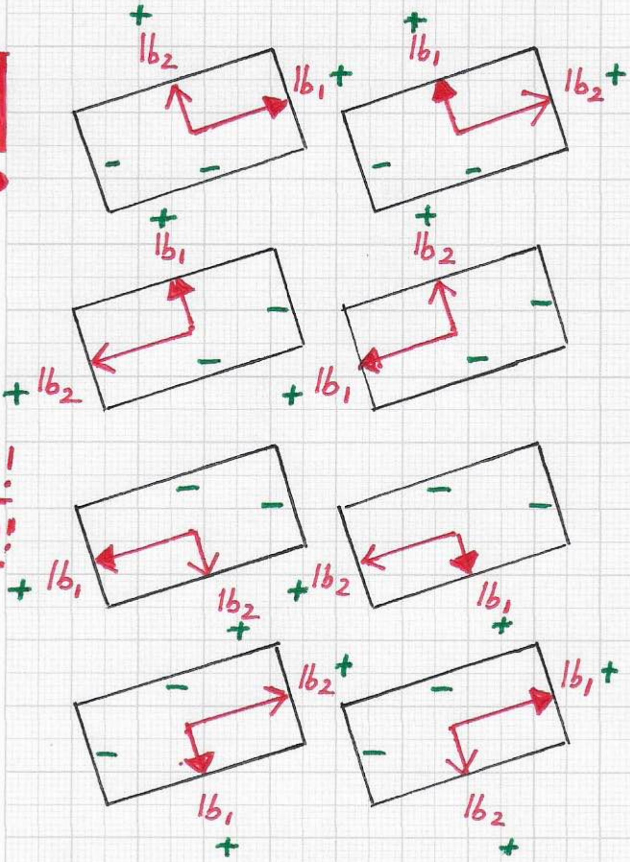
■ OBJECT AND MATERIAL EIGENFUNCTIONS - Cont'd.

• Laplacian eigenfunctions and neural networks:... The determination of an optimal set of orthonormal basis vectors $u_1, u_2, \ldots$ is motivated by the goal of defining an optimal set of orthogonal basis vectors $b_1, b_2, \ldots$ for a (hyper-) box in D-dimensional space. (The box will subsequently be used for classification decision-making calculations instead of a parallelepiped implied by non-orthogonal basis vectors.) The left figure shows the eight acceptable ways to define a pair of orthogonal basis vectors b_1 and b_2 with a 2D box.

$D=2$: $8 = 2^2 \cdot 2! = 2^D \cdot D!$



Eight possible and "equivalent pairs" of a pair of orthogonal basis vectors b_1, b_2.

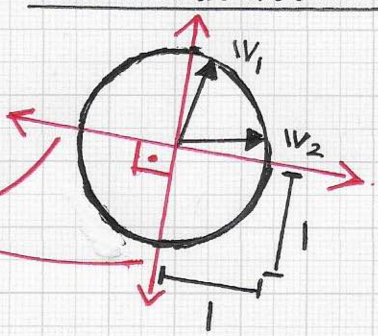
The fact that eight basis vector pairs are possible for a 2D box results from the facts that (i) four (2^2) ways exist for designating a box edge as a positive (+) or negative (−) edge and (ii) two (2!) possibilities exist to assign indices (1 or 2) to a pair of basis vectors. The table shows how rapidly $2^D \cdot D!$ values increase. • • •

D	$2^D \cdot D!$
1	2
2	8
3	48
4	384
5	3840

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

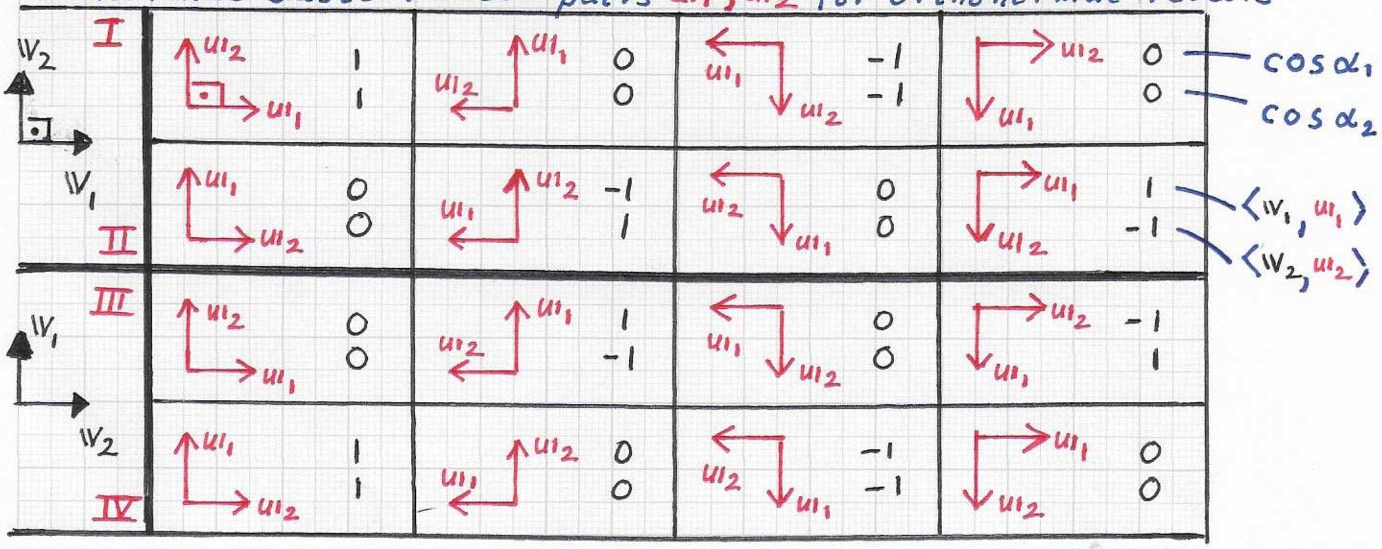
- Laplacian eigenfunctions and neural networks: ...

Orthogonal Line pair



Optimal "axis line pair" constructed for two vectors $\mathbb{w}_1$ and $\mathbb{w}_2$.

Considering the fact that one can choose one of all the combinatorially possible basis vector sets for a D-dimensional (hyper-) box, one does not need to worry about the positive or negative sign (orientation) of a basis vector or the indices used for D basis vectors; they only must be orthogonal. The left figure sketches an orthogonal line pair that, when optimally oriented (rotated) fully captures the eight equivalent and acceptable basis vector pairs for two given vectors $\mathbb{w}_1$ and $\mathbb{w}_2$. The table lists the 16 possible and acceptable orthonormal basis vector pairs $\mathbb{u}_1, \mathbb{u}_2$ for orthonormal vectors

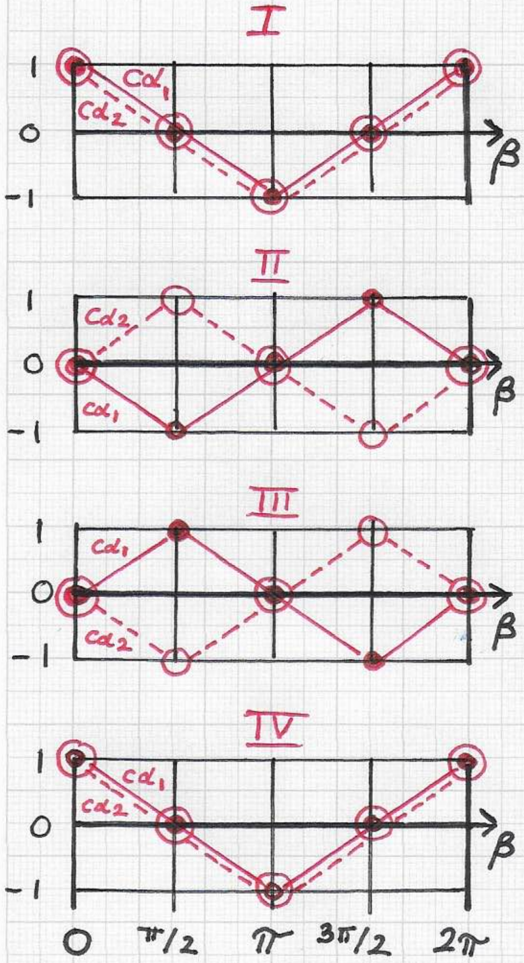


I	1, 1	0, 0	−1, −1	0, 0
II	0, 0	−1, 1	0, 0	1, −1
III	0, 0	1, −1	0, 0	−1, 1
IV	1, 1	0, 0	−1, −1	0, 0

$\mathbb{w}_1, \mathbb{w}_2$. Each entry lists the values of $\langle \mathbb{w}_1, \mathbb{u}_1 \rangle$ and $\langle \mathbb{w}_2, \mathbb{u}_2 \rangle$. ...

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:...



—— $\cos\alpha_1 = \langle \mathbb{v}_1, \mathbb{u}_1 \rangle$

- - - $\cos\alpha_2 = \langle \mathbb{v}_2, \mathbb{u}_2 \rangle$

Cases I, IV:

$$C(\beta) = \cos(\alpha_1(\beta)) + \cos(\alpha_2(\beta)) = 2c\alpha_1 = 2c\alpha_2$$

Cases II, III:

$$C(\beta) = \cos(\alpha_1(\beta)) + \cos(\alpha_2(\beta)) = 0$$

The table provided on the previous page analyzes the special case of a given pair of vectors: $\mathbb{v}_1$ and $\mathbb{v}_2$ are orthonormal — and the two possible index choices are considered. The $\mathbb{u}$-system is rotated relative to the given $\mathbb{v}$-system by rotation angles 0, $\pi/2$, π and $3\pi/2$. Regarding cases I, II, III and IV, the table lists for each of these cases, and each rotation angle, the resulting values of $\cos\alpha_1$ and $\cos\alpha_2$. The left figure shows the same information by providing approximating poly-line plots that use straight line segments for connecting two consecutive cos-values, terminating the plots at the value $\beta = 2\pi$ due to the periodicity of the cos function.

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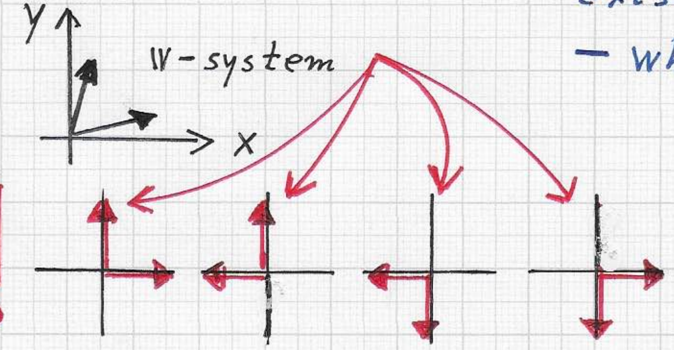
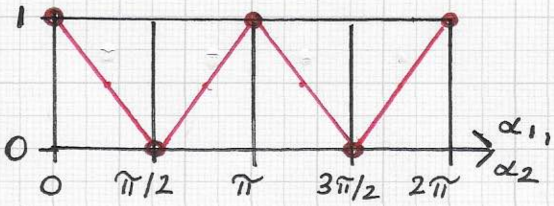
▪ OBJECT AND MATERIAL EIGENFUNCTIONS - Cont'd.

• Laplacian eigenfunctions and neural networks: ...



Given W-system and its resulting four optimal ω-systems.

For a given W-system, there exist four optimal ω-systems — when ignoring the issue of all possible ways to assign indices to the vectors of a pair of vectors. The left figure provides a sketch. The goal is to formulate and solve this optimization in an algebraic way that yields these four optimal solutions.



The squared functions $\cos^2(\alpha_1)$ and $\cos^2(\alpha_2)$. They have the same extrema.

• Note. The left figure shows qualitative sketches of the graphs of the functions $c^2\alpha_1 = \cos^2\alpha_1 = (\cos\alpha_1)^2$ and $c^2\alpha_2 = \cos^2\alpha_2 = (\cos\alpha_2)^2$ — which are equal for the considered example. The first derivative of $\cos^2\alpha_1$ is the function $-2\cos\alpha_1 \cdot \sin\alpha_2$, which has value 0 at $0, \pi/2, \pi, 3\pi/2, 2\pi$ in the interval $[0, 2\pi]$. For these α_1-values, one obtains the allowable four possible optimal solutions shown in the top figure. (The same holds for α_2.) This optimization problem is subject to the constraint $x^2+y^2=1$ concerning all vectors; Lagrange multipliers can be used.

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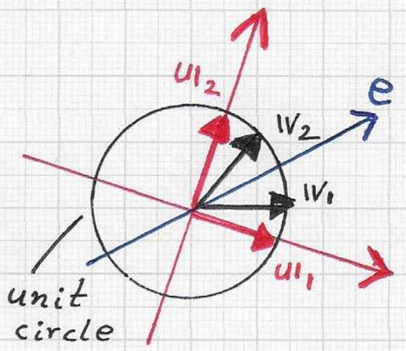
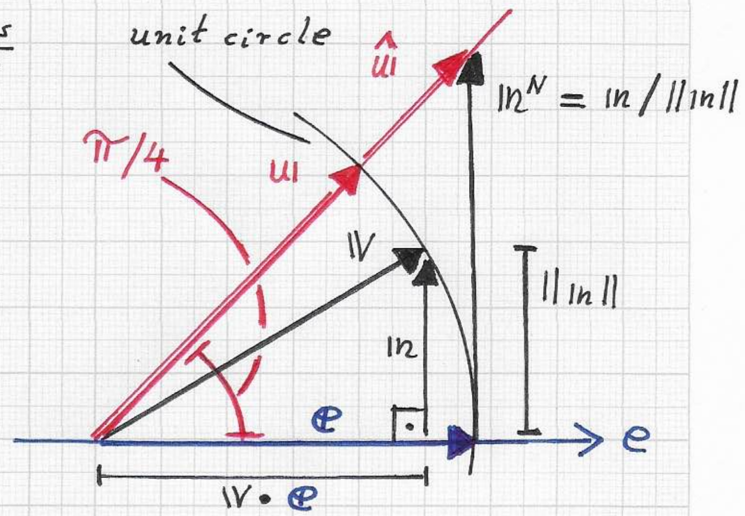
OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

- Laplacian eigenfunctions and neural networks: ...





- **Note.** The top-left figure shows the desired transformation – mapping two non-orthogonal vectors $\mathbb{v}_1$ and $\mathbb{v}_2$ in an "optimal" way to two orthogonal vectors $\mathbb{u}_1$ and $\mathbb{u}_2$. The top-right figure illustrates how one can solve this transformation problem in the 2D case, using a geometrical approach. For simplicity, the figure and formulas do not use the indices of the $\mathbb{v}$- and $\mathbb{u}$-vectors. The "e-direction" and vector $\mathbb{e}$ can be understood as "average" or "eigen" direction implied by $\mathbb{v}_1$ and $\mathbb{v}_2$.

$$\mathbb{n} = \mathbb{v} - (\mathbb{v} \cdot \mathbb{e})\,\mathbb{e}$$

$$\hat{\mathbb{u}} = \mathbb{e} + \mathbb{n}^N$$

$$\mathbb{u} = \hat{\mathbb{u}} / \|\hat{\mathbb{u}}\| = \hat{\mathbb{u}} / \sqrt{2}$$

$$\left(\|\hat{\mathbb{u}}\| = \sqrt{2} : \ \|\mathbb{e}\|^2 + \|\mathbb{n}^N\|^2 = 2 \ \Rightarrow \|\hat{\mathbb{u}}\|^2 = 2 \right)$$

$$\|\mathbb{n}\|^2 + (\mathbb{v} \cdot \mathbb{e})^2 = 1$$

$$\Rightarrow \|\mathbb{n}\| = {}^{+}\!/_{-} \sqrt{1 - (\mathbb{v} \cdot \mathbb{e})^2}$$

$$\Rightarrow \boxed{\begin{aligned} \mathbb{u} &= \hat{\mathbb{u}} / \sqrt{2} = \left(\mathbb{e} + \mathbb{n}^N \right) / \sqrt{2} = \\ &= \left(\mathbb{e} + \mathbb{n} / \|\mathbb{n}\| \right) / \sqrt{2} \\ &= \left(\mathbb{e} \ {}^{+}\!/_{-} \frac{\mathbb{v} - (\mathbb{v} \cdot \mathbb{e})\,\mathbb{e}}{\sqrt{1 - (\mathbb{v} \cdot \mathbb{e})^2}} \right) / \sqrt{2} \end{aligned}}$$

...