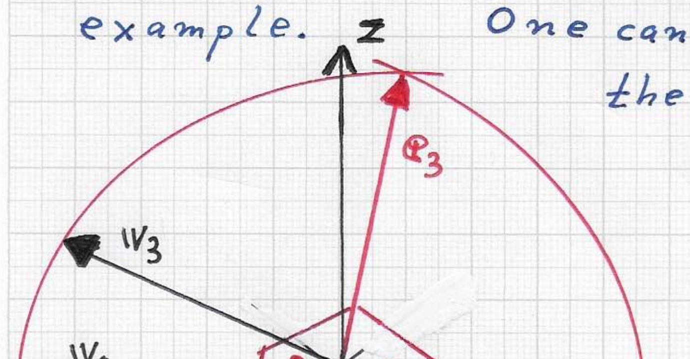
OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:… The Note included on the previous page is relevant: In the 2D setting the "duality" between an orthonormal Φ-eigensystem and the needed "optimal" $\mathbb{U}$-system, defined by orthonormal vectors $\mathbb{U}_1$ and $\mathbb{U}_2$, is obvious. Therefore, it is of interest to explore whether this fact can be exploited in the general D-dimensional setting. The three orthonormal eigenvectors Φ_1, Φ_2 and Φ_3 provided on the previous page concern a specific 3D example. One can view the calculation of the needed $\mathbb{U}$-system as an optimization problem – a "morphing" problem – that is concerned with the mapping of the PCA-generated Φ-eigensytem to the given set of normalized, non-orthogonal set $\{\mathbb{V}_1, \mathbb{V}_2, \mathbb{V}_3\}$ – minimizing "system distance" and requiring minimal "mapping energy," see left figure. …

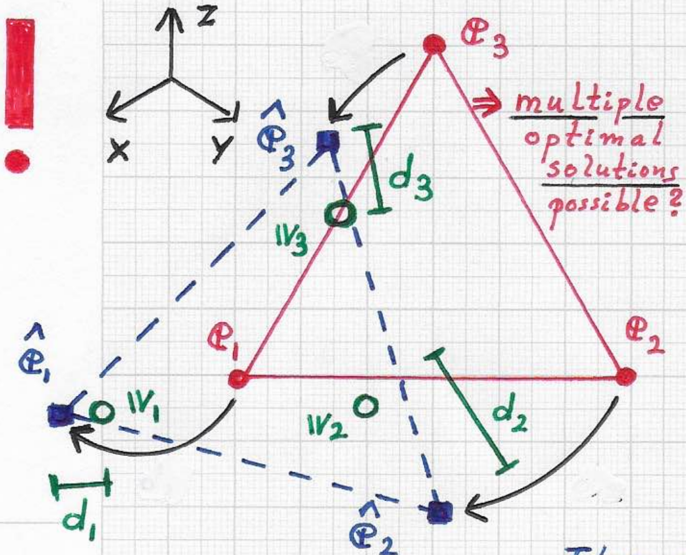


$\mathbb{V}_1 = (\sqrt{3}/2, -1/2, 0)^T$, $\Phi_1 = (.96, 0, .29)^T$,
$\mathbb{V}_2 = (\sqrt{3}/2, 1/2, 0)^T$, $\Phi_2 = (0, 1, 0)^T$,
$\mathbb{V}_3 = (\sqrt{2}/2, 0, \sqrt{2}/2)^T$; $\Phi_3 = (-.29, 0, .96)^T$

A minimal energy mapping for morphing the Φ- to the $\mathbb{V}$-system must be defined.

OBJECT AND MATERIAL EIGENFUNCTIONS — Cont'd.

• Laplacian eigenfunctions and neural networks: ...



One can understand the mapping of the PCA-generated orthonormal eigensystem $\{\mathbb{e}_1, \mathbb{e}_2, \mathbb{e}_3\}$ to the desired "optimal" orthonormal system $\{\hat{\mathbb{e}}_1, \hat{\mathbb{e}}_2, \hat{\mathbb{e}}_3\}$ as a mapping that leads to an "optimal" placement of the vectors $\hat{\mathbb{e}}_1$, $\hat{\mathbb{e}}_2$ and $\hat{\mathbb{e}}_3$ such that "distance" is minimized. The left figure illustrates this mapping for the considered 3D example, showing the involved unit vectors in a 2D projection plane as points •, ○ or ■. The parameters d_1, d_2 and d_3 represent a measure for "distance" between the given vectors $\mathbb{v}_i$ and the "optimal" vectors $\hat{\mathbb{e}}_i$. For example, one could consider the value of the sum of the squared "distances" d_i to be minimized, i.e., $\sum_{i=1}^{3} d_i^{\,2} \rightarrow MIN$.

One can also view this mapping as a mapping that maps the orthonormal matrix with column vectors $\mathbb{e}_1$, $\mathbb{e}_2$ and $\mathbb{e}_3$ to the orthonormal matrix with column vectors $\hat{\mathbb{e}}_1$, $\hat{\mathbb{e}}_2$ and $\hat{\mathbb{e}}_3$, subject to leading to a best (matrix) approximation of the matrix with non-orthogonal column vectors $\mathbb{v}_1$, $\mathbb{v}_2$ and $\mathbb{v}_3$.

...

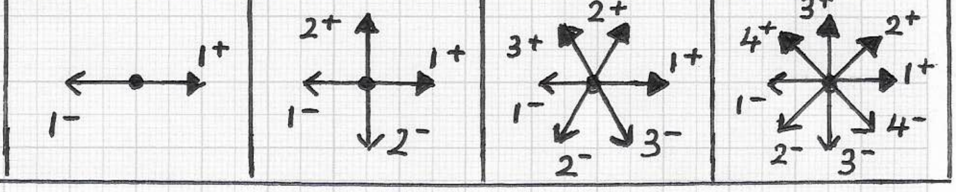
OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... Concerning the 3D case, one can define the approximation problem also as follows: Given unit vectors $\mathbb{v}_1$, $\mathbb{v}_2$ and $\mathbb{v}_3$ (not mutually orthogonal to each other), determine unit and mutually orthogonal vectors $\hat{\mathbb{e}}_1$, $\hat{\mathbb{e}}_2$ and $\hat{\mathbb{e}}_3$ that approximate the given vectors in a best-approximation sense, i.e., in an error-minimization sense. Another possible problem definition is using matrices: Given a 3-by-3 matrix V with unit column vectors $\mathbb{v}_1$, $\mathbb{v}_2$ and $\mathbb{v}_3$, i.e., $V = [\mathbb{v}_1\ \mathbb{v}_2\ \mathbb{v}_3]$, determine a 3-by-3 matrix $\mathbb{E}$ with unit and mutually orthogonal vectors $\hat{\mathbb{e}}_1$, $\hat{\mathbb{e}}_2$ and $\hat{\mathbb{e}}_3$, i.e., $\mathbb{E} = [\hat{\mathbb{e}}_1\ \hat{\mathbb{e}}_2\ \hat{\mathbb{e}}_3]$, such that $\mathbb{E}$ is a best matrix approximation of V – minimizing a chosen measure for the "distance" between V and $\mathbb{E}$, i.e., "distance$(V,\mathbb{E})$ = MIN." When understanding this problem as an approximation problem, several related topics and solution approaches come to mind: (i) least-squares approximation and normal equations; (ii) orthogonal, orthonormal and rotation matrices; (iii) best approximation with constraints and method of Lagrange multipliers; (iv) approximation with energy-minimizing splines, e.g., thin-plate spline (TPS) mappings/deformations; (v) linear equation system resulting from minimization of quadratic error. ...

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

- **Laplacian eigenfunctions and neural networks:...** The optimal orthonormal coordinate system for a given set of linearly independent unit (but not mutually orthogonal) vectors $\mathbb{v}_1, \ldots, \mathbb{v}_D$ is not unique. The optimization problem concerned with the determination of orthonormal coordinate system basis vectors has as its solution a finite set, of sets of D vectors each, that defines allowable, equally optimal coordinate system basis vector solutions. This fact is a consequence of the possibility to accept either the "positive" (+) or "negative" (−) direction/orientation of each basis vector. In the following, we discuss this aspect in detail for low-dimensional cases that are helpful for understanding the general D-dimensional setting. It is possible to analyze this issue purely from a combinatorial perspective. The figure shown here

D	0	1	2	3	4
basis vectors –"projections"	•				

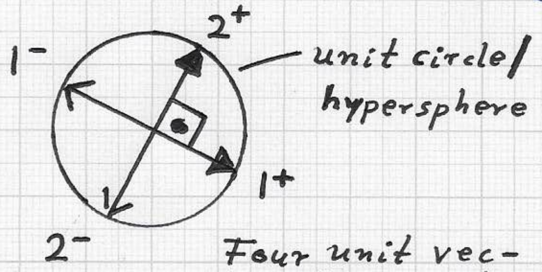


summarizes in an abstract way the unit vector sets — for dimensions $D = 0, 1, 2, 3, 4$ — from which one can/must select D to define an allowable basis.

...

OBJECT AND MATERIAL EIGENFUNCTIONS - Cont'd.

• Laplacian eigenfunctions and neural networks: ...



D=2

Four unit vectors from which one must select 2.

Thus, one must pick a subset consisting of **D** unit and mutually orthogonal vectors from the set of candidate vectors, called in this discussion $1^+, 2^+, \ldots, D^+, 1^-, 2^-, \ldots, D^-$. The set of candidate vectors contains **2D** unit vectors, and every subset consisting of **D** mutually orthogonal unit vectors is an optimal set of coordinate system basis vectors. By definition, an optimal set never contains both the vector d^+ and d^-, $d \in \{1, 2, \ldots, D\}$, as $d^- = -d^+$. The top-left figure illustrates the **2**-dimensional case. From the combinatorics perspective, one is interested in all allowable optimal solutions and the number of them. The

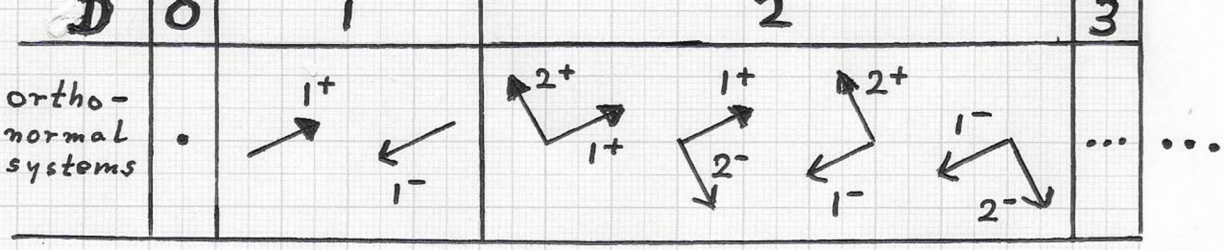


figure included here shows the possible optimal orthonormal basis vector sets for dimensions **D**=0, 1, 2. For example, for **D=2** one pair of orthogonal lines defines the shown four optimal solutions, resulting from the two possible orientations (+, -) for each line. • • •