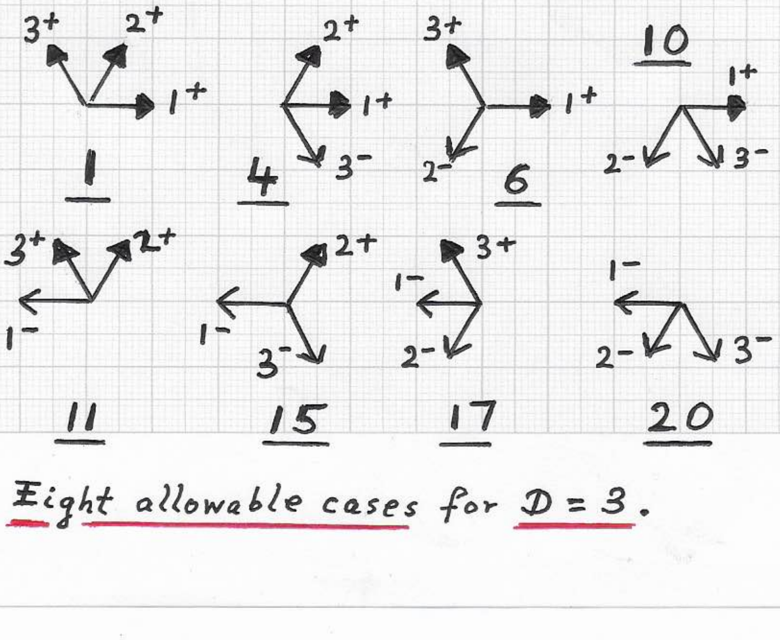
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... We discuss the cases $D = 3$ and $D = 4$ in detail in the following, to gain deeper insight into the sets of optimal basis vectors – from a combinatorics viewpoint. For $D = 3$, we must select three orthonormal vectors from $2D = 6$ candidates. Thus, there exist $\binom{6}{3} = 20$ combinatorially possible cases. These cases (1–20) are listed in the left table.

case	1^+	2^+	3^+	1^-	2^-	3^-	Y/n
1	•	•	•				Y
2	•	•		•			n
3	•	•			•		n
4	•	•				•	Y
5	•		•	•			n
6	•		•		•		Y
7	•		•			•	n
8	•			•	•		n
9	•			•		•	n
10	•				•	•	Y
11		•	•	•			Y
12		•	•		•		n
13		•	•			•	n
14		•		•	•		n
15		•		•		•	Y
16		•			•	•	n
17			•	•	•		Y
18			•	•		•	n
19			•		•	•	n
20				•	•	•	Y

Geometrically, only those possibilities are allowable that lead to sets of three vectors that do not simultaneously include the "+ and – versions" of a vector. As a consequence of this mutual-orthoganality requirement, the allowable vector sets are only the eight sets shown in the left figure. • • •



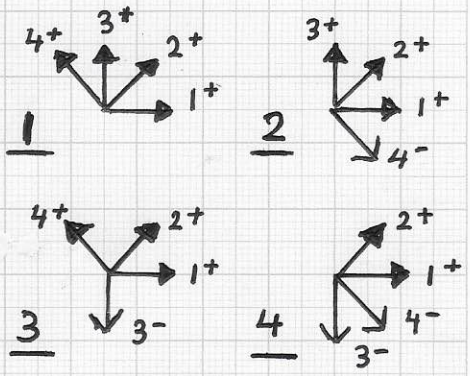
Eight allowable cases for $D = 3$.

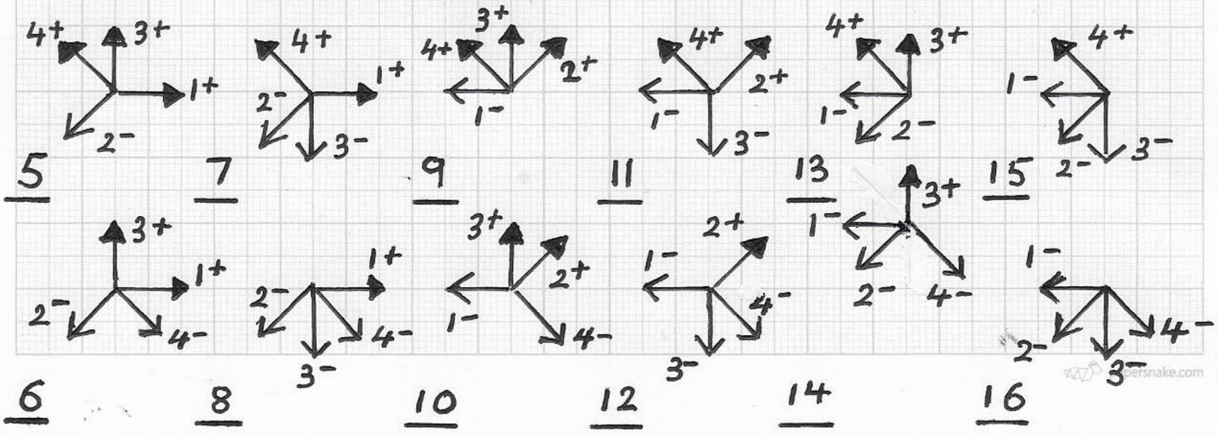
■ OBJECT AND MATERIAL EIGENFUNCTIONS - Cont'd.

• Laplacian eigenfunctions and neural networks:... Via induction one sees that of the combinatorially possible $\binom{2D}{D}$ sets of D unit vectors in D-dimensional space 2^D sets are geometrically allowable sets of orthonormal vectors. For $D=4$, $2^4=16$ sets define equally optimal, allowable coordinate system basis vector sets. We provide the case table and illustrations of these 16 cases for the 4-dimensional setting.

The left table shows all **16** allowable possibilities. These are sketches:

case	1+	2+	3+	4+	1−	2−	3−	4−
1	•	•	•	•				
2	•	•	•					•
3	•	•		•			•	
4	•	•					•	•
5	•		•	•		•		
6	•		•			•		•
7	•			•		•	•	
8	•					•	•	•
9		•	•	•	•			
10		•	•		•			•
11		•		•	•		•	
12		•			•		•	•
13			•	•	•	•		
14			•		•	•		•
15				•	•	•	•	
16					•	•	•	•





"Projections" of the **16** allowable cases for $D=4$. ...

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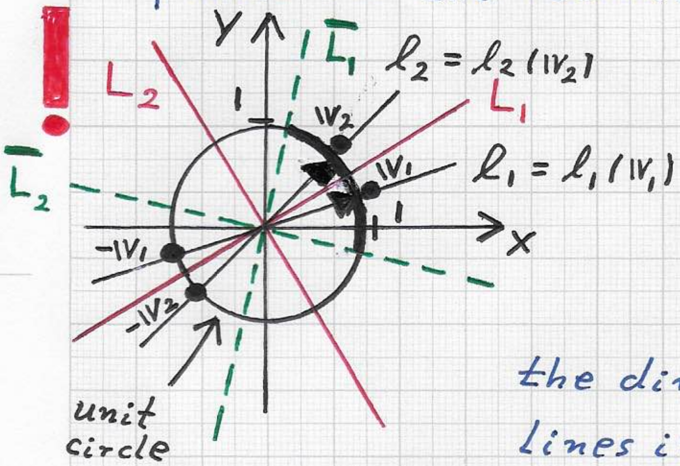
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... The detailed discussion of the sets of optimal and allowable coordinate system basis vectors for $D \leq 4$ shows that 2^D sets define the candidate orthonormal vector sets one can select from. For the reduction of the problem to a purely geometrical problem, one could define the problem in terms of only line geometry (and ignore, at least initially, the direction/orientation of the lines involved), see left figure.



It is now possible to define the problem as follows (for $D=2$): Given two vectors $\mathbb{v}_1$ and $\mathbb{v}_2$ satisfying the conditions $\langle \mathbb{v}_1, \mathbb{v}_2 \rangle \neq 0$ and $|\langle \mathbb{v}_1, \mathbb{v}_2 \rangle| \neq 1$, consider the two non-orthogonal lines $\ell_1 = \ell_1(\mathbb{v}_1)$ and $\ell_2 = \ell_2(\mathbb{v}_2)$; apply principal components analysis (PCA) to ℓ_1 and ℓ_2, generating to eigenvalues and two orthogonal eigenvectors for the point/positional vector data set $\{\mathbb{v}_1, \mathbb{v}_2, -\mathbb{v}_1, -\mathbb{v}_2\}$ and consequently the two orthogonal lines L_1 and L_2; calculate the desired dual line pair, i.e., the two orthogonal lines $\bar{L}_1$ and $\bar{L}_2$, via PCA or Lagrange multipliers, for example. • • •

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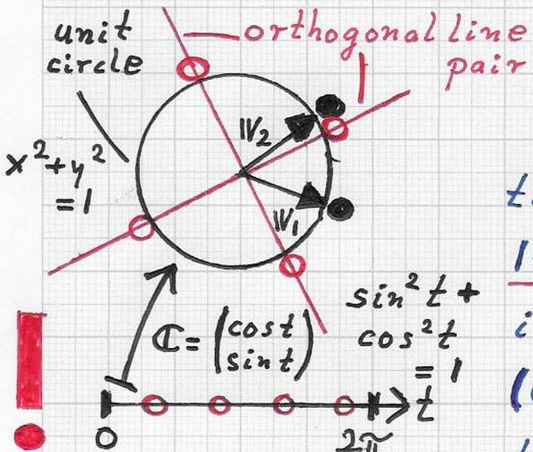
OBJECT AND MATERIAL EIGENFUNCTIONS - Cont'd.

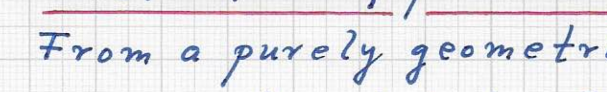
- Laplacian eigenfunctions and neural networks:...



Parametric optimization. When treating the described problem as a "parametric optimization problem," one views the problem as optimization on a 1-dimensional (1-manifold) domain, i.e., the unit circle with center $(0,0)^T$. A parametrization of the unit circle is, for example,

$$\mathbb{C}(t) = (x(t), y(t))^T = (\cos t, \sin t)^T, \ t \in [0, 2\pi].$$

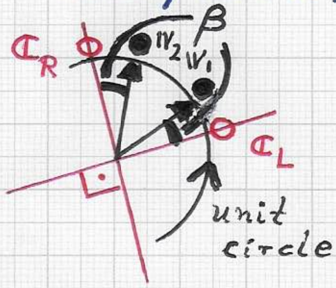
From a purely geometrical viewpoint, the problem can now be stated as follows: Given two points '•' on the unit circle (defined by two unit non-orthogonal vectors $\mathbb{V}_1$ and $\mathbb{V}_2$, where $\mathbb{V}_1 \neq +/- \mathbb{V}_2$), determine a pair of orthogonal lines (intersecting at the origin) with an orientation such that the set of four equidistantly spaced intersection points includes a pair of two neighboring intersection points 'o' that optimally approximates the given two points '•'. The top-left figure sketches the problem statement.

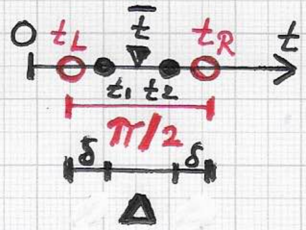


One can design one solution approach based on the parameter (t) domain: The points '•' have associated t-values. The points 'o' have associated t-values. Define the 'o' t-values such that the '•' pair of t-values lies in the middle of a 'o' t-value pair....

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... The bottom-left figure on the previous page shows a specific t-value configuration that is not optimal. One must therefore determine a placement of the four 'o' t-values in the interval $[0, 2\pi]$ that includes a pair of neighboring 'o' t-values that has as its average t-value the average of the two given, fixed '•' t-values. Of course, one must consider the periodic nature of the setting, i.e., $\mathbb{C}(t) = \mathbb{C}(t \pm k \cdot 2\pi)$, $k \in \mathbb{N}$. The solution approach is sketched in the left figure. The figure shows an optimal placement of 'o' t-values. One can intuitively understand that the following calculations must be done when the two points $\mathbb{V}_1$ and $\mathbb{V}_2$ with associated parameter values t_1 and t_2 with average $\bar{t} = \frac{t_1 + t_2}{2}$ are given:





Optimal parameter value placement. $\bar{t} = \frac{t_1 + t_2}{2}$, $t_R - t_L = \pi/2$.

$$t_L = \bar{t} - \pi/4 \quad , \quad t_R = \bar{t} + \pi/4 \ ;$$

$$\mathbb{C}_L = \mathbb{C}(t_L) = \left(\cos(\bar{t} - \pi/4), \sin(\bar{t} - \pi/4)\right)^T,$$

$$\mathbb{C}_R = \mathbb{C}(t_R) = \left(\cos(\bar{t} + \pi/4), \sin(\bar{t} + \pi/4)\right)^T.$$

The points/positional vectors $\mathbb{C}_L$ and $\mathbb{C}_R$ are orthogonal:

$$\langle \mathbb{C}_L, \mathbb{C}_R \rangle = \cos(\bar{t} - \pi/4)\cos(\bar{t} + \pi/4) + \sin(\bar{t} - \pi/4)\sin(\bar{t} + \pi/4)$$

$$= \cos(\bar{t} - \pi/4 - \bar{t} - \pi/4) = \cos(-\pi/2) = 0 \Rightarrow \mathbb{C}_L \perp \mathbb{C}_R .$$

(Here, the angle theorem $\cos\alpha \cos\beta + \sin\alpha \sin\beta = \cos(\alpha - \beta)$ is used.) •••