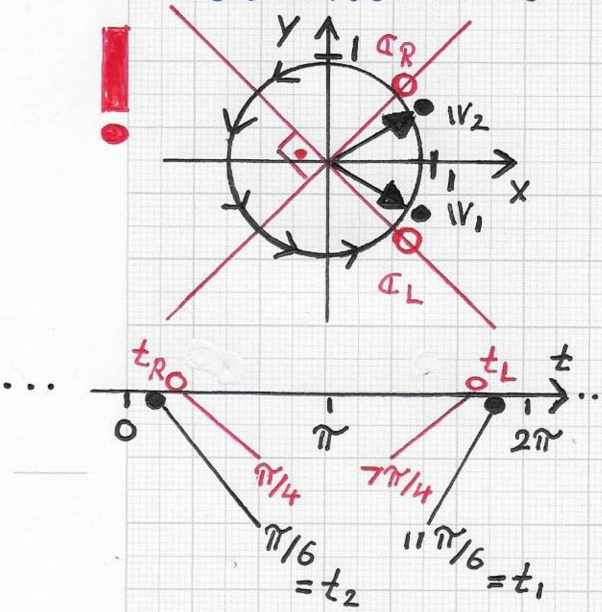
Stratovan

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

- Laplacian eigenfunctions and neural networks: … We consider a simple case that shows periodic/cyclic behavior that one must consider. The two given points/positional vectors are

$\mathbb{V}_1 = (\sqrt{3}/2, -1/2)^T$, $\mathbb{V}_2 = (\sqrt{3}/2, 1/2)^T$.



Simple 2D example.

Since a point on the unit circle is defined as $\mathbb{C}(t) = (\cos t, \sin t)^T$, we obtain the parameter values t_1 and t_2 of $\mathbb{V}_1$ and $\mathbb{V}_2$ via the arccos and arcsin functions:

$\arccos(\sqrt{3}/2) = \pi/6$ and

$\arcsin(+/- 1/2) = +/- \pi/6$

$(\Rightarrow \arcsin(-1/2) = 11\pi/6)$.

Thus, one determines that $t_1 = 11\pi/6$ and $t_2 = \pi/6$, when using only t-values in the interval $[0, 2\pi]$. Consequently, the needed values are:

$\bar{t} = 0$ (alternatively $\bar{t} = 2\pi$),

$t_L = 0 - \pi/4 = -\pi/4$ (alternatively $t_L = 2\pi - \pi/4 = 7\pi/4$),

$t_R = 0 + \pi/4 = \pi/4$ (alternatively $t_R = 2\pi + \pi/4 = 9\pi/4$).

The resulting pair of optimal orthonormal vectors $\mathbb{C}_L$ and $\mathbb{C}_R$ that (optimally) approximates the given unit non-orthogonal pair $\mathbb{V}_1$ and $\mathbb{V}_2$ is

$\mathbb{U}_1 = \mathbb{C}_L = (\cos(7\pi/4), \sin(7\pi/4))^T = (\sqrt{2}/2, -\sqrt{2}/2)^T$,

$\mathbb{U}_2 = \mathbb{C}_R = (\cos(\pi/4), \sin(\pi/4))^T = (\sqrt{2}/2, \sqrt{2}/2)^T$.

Summary: Subtracting/adding $\frac{\pi}{4}$ from/to $\bar{t}$ defines the pair $\mathbb{C}_L$ and $\mathbb{C}_R$.

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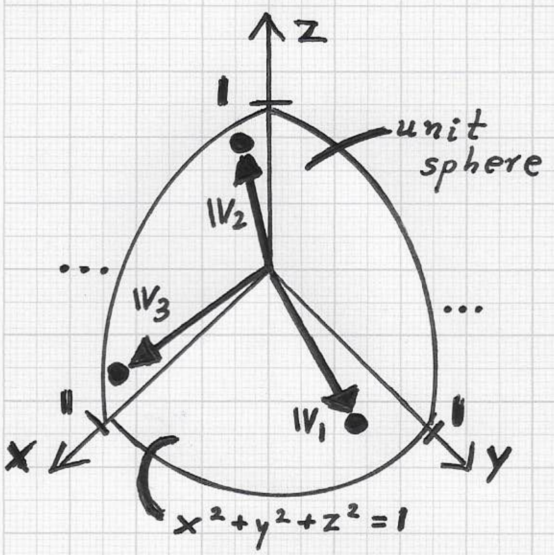
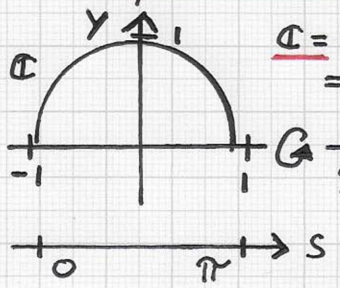
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

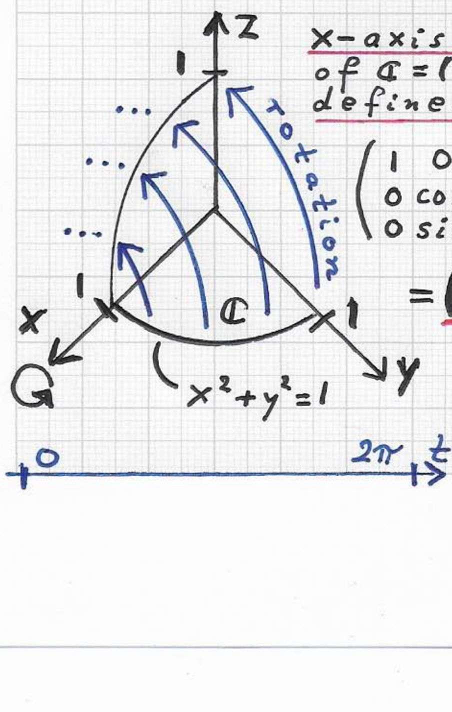
• Laplacian eigenfunctions and neural networks: ...



Three non-orthogonal unit vectors $\mathbb{v}_1$, $\mathbb{v}_2$ and $\mathbb{v}_3$ – postional vectors of '●'.



$$\mathbb{C} = \mathbb{C}(s) = (x(s), y(s))^T = \begin{pmatrix} \cos(s) \\ \sin(s) \end{pmatrix}, \quad s \in [0, \pi]$$



x-axis rotation of $\mathbb{C} = (\cos(s), \sin(s), 0)^T$ defines unit sphere:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(t) & -\sin(t) \\ 0 & \sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} \cos(s) \\ \sin(s) \\ 0 \end{pmatrix} = (\cos(s), \sin(s)\cos(t), \sin(s)\sin(t))^T,$$

$$s \in [0, \pi], \; t \in [0, 2\pi]$$

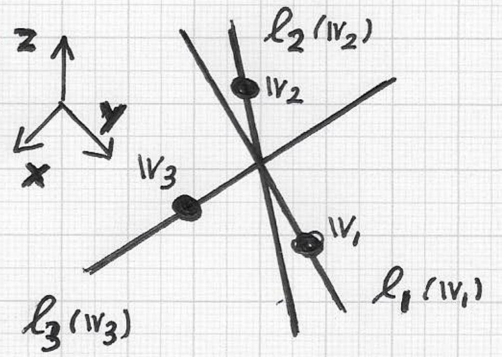
Parametric definition of unit sphere via semi-circle rotation.

Since the discussion of the 2D case uses a parametric representation of the unit circle, the initial treatment of the 3D case is based on a parametric definition as well – by considering a bivariate parametric formula for points on the unit sphere $x^2+y^2+z^2=1$. The top-left figure illustrates the given point/vector data for which an optimal set of three unit orthogonal point/vector data must be determined. The set $\{\mathbb{v}_1, \mathbb{v}_2, \mathbb{v}_3\}$ is given, defining three non-orthogonal lines intersecting at the origin that must be approximated optimally by an orthogonal set $\{\mathbb{u}_1, \mathbb{u}_2, \mathbb{u}_3\}$, defining three orthogonal lines.

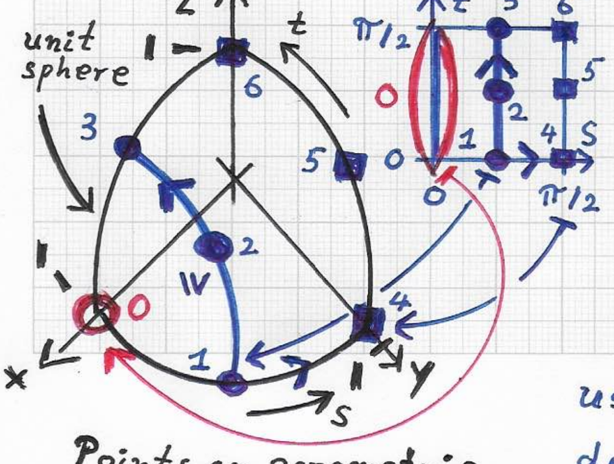
...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... The two bottom-left figures on the previous page shows one possibility to define the unit sphere parametrically: First, one defines a semi-circle in the xy-plane as shown, for example; second, one rotates this semi-circle with the x-axis serving as rotation axis, generating the complete unit sphere. (This particular parametrization leads to the two poles $(+/-1,0,0)^T$, as $(s=0,t) \mapsto (1,0,0)^T$ and $(s=\pi,t) \mapsto (-1,00)^T$, for $t \in [0,2\pi]$.) Once again, one can view the given data as three lines $\ell_1(\mathbb{V}_1)$, $\ell_2(\mathbb{V}_2)$ and $\ell_3(\mathbb{V}_3)$ that intersect in one point, see left figure. The chosen parametric representation of the sphere $x^2+y^2+z^2=1$ implies that one must ultimately perform computationally expensive trigonometric functions. For example, the parameter values s and t of a given point $\mathbb{V}=(x,y,z)^T$ are $s=\arccos(x)$, $t=\arccos(y)=\arcsin(z)$, using the st-parametrization as defined on the previous page. ...



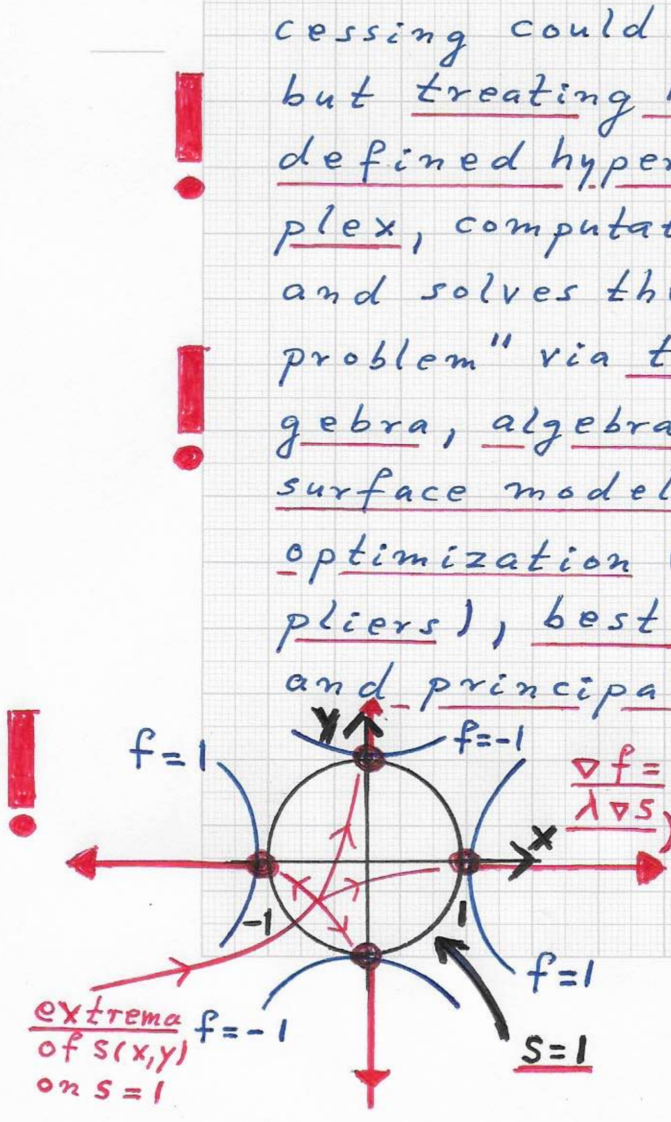
Given non-orthogonal lines defined by points.



Points on parametric sphere.

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... While parametric representations are commonly used in differential geometry – for the definition and calculation of differential properties of curves and surfaces – they do not seem to be advantageous for our purposes, i.e., performing computations involving unit hyper-spheres in D-dimensional space. Keeping in mind concepts used for parametric surface processing could be helpful for our application, but treating hyper-spheres as implicitly defined hyper-surfaces might be less complex, computationally. Thus, one investigates and solves the optimal basis vector "design problem" via techniques from geometry, algebra, algebraic geometry, implicit (hyper-) surface modeling, constrained algebraic optimization (including Lagrange multipliers), best approximation via least squares and principal component analysis (PCA).



We discuss a simple 2D, bivariate example to perform constrained optimization via the Lagrange-multiplier method: Calculate the extrema of $f(x,y) = x^2 - y^2$ on the contour $s=1$ of $s(x,y) = x^2 + y^2$, see left figure. • • •

OBJECT AND MATERIAL EIGENFUNCTIONS - Cont'd.

• Laplacian eigenfunctions and neural networks:... • Note. A reference that provides a high-level summary of constrained optimization and Lagrange multipliers is "Vector Calculus" by J. E. Marsden and A. J. Tromba, Freeman and Company.

The example sketched on the previous page regards the problem of finding the extrema of the function $f(x,y) = x^2 - y^2$ on the unit circle (hyper-sphere) defined implicitly, i.e., $s(x,y) = x^2 + y^2 = 1$. Based on the Lagrange-multiplier method, the extrema of f on $s=1$ are points on the circle where $\nabla f = \lambda \nabla s$. Here, $\nabla f = (2x, -2y)$ and $\nabla s = (2x, 2y)$. Thus, f is extremal at points on the circle where $(2x, -2y) = \lambda (2x, 2y)$. The resulting equations are: (i) $x^2+y^2=1$; (ii) $2x = \lambda 2x$; (iii) $-2y = \lambda 2y$. The following cases must be considered:

- (ii) $2x = \lambda 2x \Rightarrow x = 0$ (or $\lambda = 1$) $\underset{(i)}{\Rightarrow} y^2 = 1 \Rightarrow y = +/-1$;
- (iii) $-2y = \lambda 2y \Rightarrow y = 0$ (or $\lambda = -1$) $\underset{(i)}{\Rightarrow} x^2 = 1 \Rightarrow x = +/-1$.

Thus, the four points $(0,-1)^T$, $(0,1)^T$, $(-1,0)^T$ and $(1,0)^T$ are the locations on the circle where f is extremal. IN ORDER TO DETERMINE THE TYPE OF EACH OF THESE FOUR (CANDIDATE) POINTS, ONE MUST USE AN ADDITIONAL TEST, E.G., THE "BORDERED HESSIAN DETERMINANT" TEST. ...