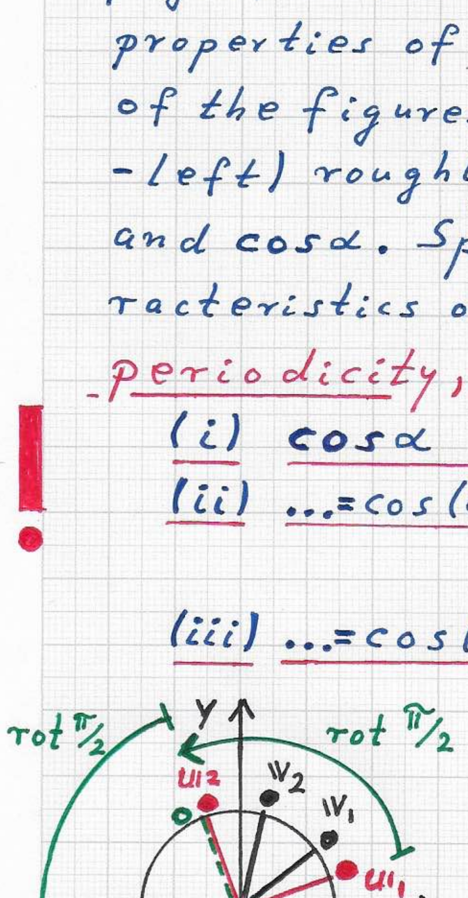
OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• **Laplacian eigenfunctions and neural networks:…** Before considering the cost functions C_1 and C_2 (previous page), it seems appropriate to recall relevant properties of trigonometric functions. One of the figures on the previous page (middle-left) roughly sketches the functions $\sin\alpha$ and $\cos\alpha$. Specifically relevant are the characteristics of the $\cos\alpha$ function, regarding periodicity, symmetry and sign behavior:

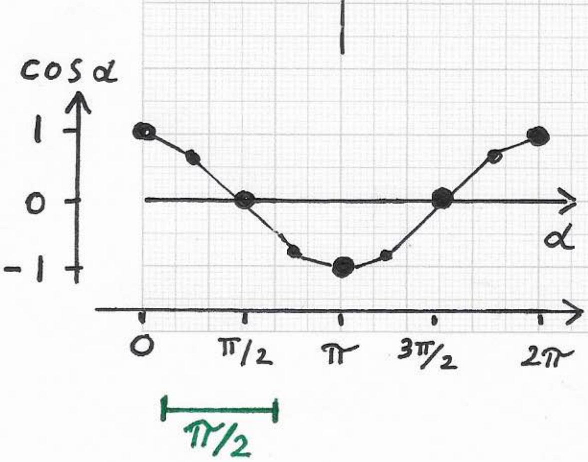
(i) $\cos\alpha = \cos(-\alpha)$ (symmetry)

(ii) $\ldots = \cos(\alpha - 2\pi) = \cos\alpha = \cos(\alpha + 2\pi) = \ldots$ (periodicity)

(iii) $\ldots = \cos(\alpha - \pi) = -\cos\alpha = \cos(\alpha + \pi) = \ldots$ (sign, periodicity)



These relationships are important to keep in mind, as they must be considered when computing and using the scalar products $u_i \cdot v_j$, with $i, j \in \{1, 2\}$, since $u_i \cdot v_j = \cos\measuredangle(u_i, v_j)$. Furthermore, the table on the previous page show that the values of the chosen cost functions are $2, 0, 2, 0, 2, 0, \ldots$, i.e., they repeat after $\pi/2$. The left figures point out that it therefore suffices to rotate (u_1, u_2) by $\pi/2$ only.



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OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... In other words, given the vector pair $\mathbb{v}_1$ and $\mathbb{v}_2$, one can define an arbitrary orthonormal vector pair $\mathbb{u}_1$ and $\mathbb{u}_2$ – their "initial configuration." It is sufficient to rotate this "initial configuration" by rotation angles $\alpha \in [0, \pi/2)$. For one angle $\alpha \in [0, \pi/2)$, one obtains one of the four "equally optimal" orthonormal vector pairs sketched on page 20 (9/16/2023) for $D = 2$ in the bottom figure. "A BEST APPROXIMATION OF THE GIVEN VECTOR PAIR $\mathbb{v}_1$ AND $\mathbb{v}_2$ IS OBTAINED BY ROTATING ANY INITIAL ORTHONORMAL VECTOR PAIR $\mathbb{u}_1$ AND $\mathbb{u}_2$ BY AN OPTIMAL ANGLE $\alpha \in [0, \pi/2)$. We now consider the suggested cost function $C_1 = (a^2 - b^2)^2 + (B^2 - A^2)^2$ in detail. First, one can consider simplifying the implicit circle representation $x^2 + y^2 = 1$ to the explicit circle representation $y = \sqrt{1 - x^2}$, $-1 \leq x \leq 1$, allowing one to define the two coordinates of all points/vectors involved in the optimization problem via one variable, x. Second, we can define the initial orthonormal vector pair $\mathbb{u}_1$ and $\mathbb{u}_2$ arbitrarily. For example, we can define $\mathbb{u}_1 = (x, \sqrt{1-x^2})^T$ and $\mathbb{u}_2 = (-\sqrt{1-x^2}, x)^T$, $0 < x \leq 1$, where the variable x is used for optimization. ...

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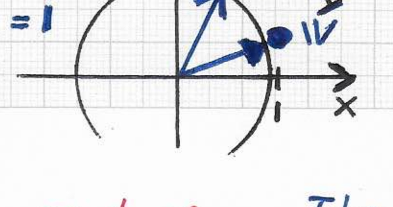
OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

- Laplacian eigenfunctions and neural networks: ... (The alternative choice for $\mathbf{u}_2$ is $\mathbf{u}_2 = (\sqrt{1-x^2}, -x)^T$.)

One can now initialize the value of x with any value in the allowed, considered range between zero and one; one subsequently obtains the initial orthonormal vector pair $\mathbf{u}_1$ and $\mathbf{u}_2$; and, by considering the four scalar products $\mathbf{u}_i \cdot \mathbf{v}_j$, and rotating the vector pair $\mathbf{u}_1$ and $\mathbf{u}_2$ in the angle range $[0, \pi/2)$ one obtains an extremal value(s) of $C_1 = C_1(\alpha)$ for a specific angle value(s). Further, one should also represent the given non-orthogonal vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ explicitly as positional vectors on the "upper semi-circle": These two unit vectors are $\mathbf{v}_1 = (x_1, y_1)^T$ and $\mathbf{v}_2 = (x_2, y_2)^T$; thus, they can be represented as $\mathbf{v}_1 = (x_1, \sqrt{1-x_1^2})^T$ and $\mathbf{v}_2 = (x_2, \sqrt{1-x_2^2})^T$, with $-1 \le x_1 \le 1$, $-1 \le x_2 \le 1$ — as $x_j^2 + y_j^2 = 1$, $j \in \{1, 2\}$. Keeping these definitions, simplifications and requirements in mind, one can analyze the function C_1 and its extremal behavior.



The dot product of two unit vectors has characteristics that are "equivalent" to those of the $\cos\alpha$ function, concerning periodicity, symmetry and sign. The figure shows $ROT(\mathbf{v}, +\alpha)$, mapping $\mathbf{v}$ to $\overline{\mathbf{v}}$.

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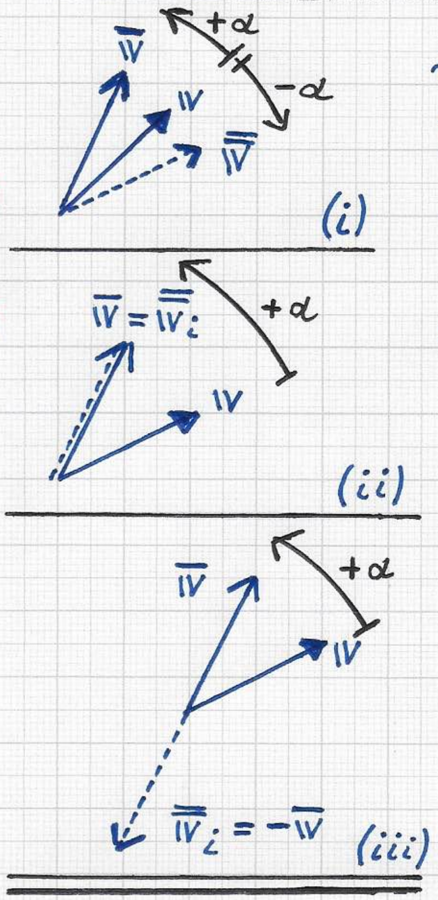
OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

- **Laplacian eigenfunctions and neural networks:...**



The dot product characteristics are illustrated geometrically by the left figures. The characteristics are:

(i) $ROT(\mathbf{v}, \alpha) = \bar{\mathbf{v}}$, $ROT(\mathbf{v}, -\alpha) = \bar{\bar{\mathbf{v}}}$

$$\Rightarrow \ldots\ \mathbf{v} \bullet \bar{\mathbf{v}} = \mathbf{v} \bullet \bar{\bar{\mathbf{v}}}$$

(ii) $ROT(\mathbf{v}, \alpha) = \bar{\mathbf{v}}$, $ROT(\mathbf{v}, \alpha + i2\pi) = \bar{\bar{\mathbf{v}}}_i$,

$$i = \ldots -2, -1, 1, 2, \ldots$$

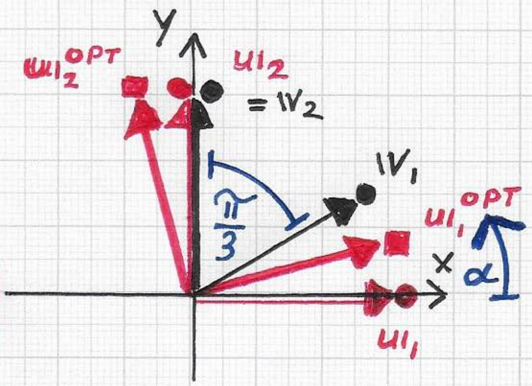
$$\Rightarrow \ldots\ \mathbf{v} \bullet \bar{\mathbf{v}} = \mathbf{v} \bullet \bar{\bar{\mathbf{v}}}_i$$

(iii) $ROT(\mathbf{v}, \alpha) = \bar{\mathbf{v}}$, $ROT(\mathbf{v}, \alpha + (2i+1)\pi) = \bar{\bar{\mathbf{v}}}_i$,

$$i = \ldots -2, -1, 0, 1, 2, \ldots$$

$$\Rightarrow \ldots\ \mathbf{v} \bullet \bar{\mathbf{v}} = -\mathbf{v} \bullet \bar{\bar{\mathbf{v}}}_i$$

These characteristics correspond to (i), (ii) and (iii) on page 11 (10/7/2023).

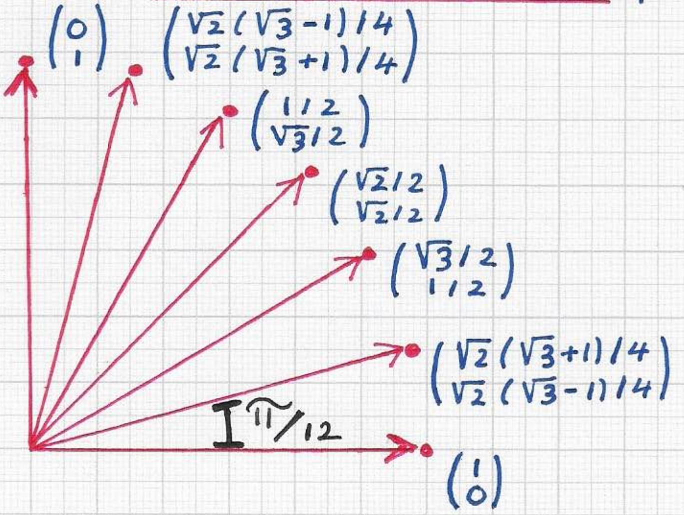


We consider the specific numerical example illustrated in the left figure. The given unit non-orthogonal vectors are $\mathbf{v}_1 = (\sqrt{3}/2, 1/2)^T$ and $\mathbf{v}_2 = (0,1)^T$. The initial values of the unit orthogonal vectors to be rotated optimally are $\mathbf{u}_1 = (1,0)^T$ and $\mathbf{u}_2 = (0,1)^T$. By visual inspection, one sees that the rotation angle $\alpha = \pi/12$ maps $\mathbf{u}_1$ and $\mathbf{u}_2$ to their optimal directions $\mathbf{u}_1^{OPT}$ and $\mathbf{u}_2^{OPT}$.

...

■ OBJECT AND MATERIAL EIGENFUNCTIONS - Cont'd.

• Laplacian eigenfunctions and neural networks:... We analyze this example by calculating dot products and and cost function values for certain rotation angles.



The left figure provides the resulting coordinate values of seven vectors, obtained by rotating the initial unit vector $(1,0)^T$ by $\alpha_i = i\cdot\pi/12$, $i=0\ldots 6$. These values are relevant for the computation of vector dot products. In the following, we use the notation u_1^i and u_2^i to refer to the vectors obtained by rotating $u_1=(1,0)^T$ and $u_2=(0,1)^T$ — i.e., the initial directions of u_1 and u_2 — by $\alpha_i = i\cdot\pi/12$, $i=0\ldots 6$. The values are:

$\alpha\,[\cdot\pi/12]$	$\cos\alpha$	u_1^i	u_2^i
0	1	$(1,0)^T$	$(0,1)^T$
1	$\sqrt{2}(\sqrt{3}+1)/4$	$(\sqrt{2}(\sqrt{3}+1)/4, \sqrt{2}(\sqrt{3}-1)/4)^T$	$(-\sqrt{2}(\sqrt{3}-1)/4, \sqrt{2}(\sqrt{3}+1)/4)^T$
2	$\sqrt{3}/2$	$(\sqrt{3}/2, 1/2)^T$	$(-1/2, \sqrt{3}/2)^T$
3	$\sqrt{2}/2$	$(\sqrt{2}/2, \sqrt{2}/2)^T$	$(-\sqrt{2}/2, +\sqrt{2}/2)^T$
4	$1/2$	$(1/2, \sqrt{3}/2)^T$	$(-\sqrt{3}/2, 1/2)^T$
5	$\sqrt{2}(\sqrt{3}-1)/4$	$(\sqrt{2}(\sqrt{3}-1)/4, \sqrt{2}(\sqrt{3}+1)/4)^T$	$(-\sqrt{2}(\sqrt{3}+1)/4, \sqrt{2}(\sqrt{3}-1)/4)^T$
6	0	$(0,1)^T$	$(-1,0)^T$

...