OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

- **Laplacian eigenfunctions and neural networks:...** Knowing the exact coordinates of $\mathbf{u}_1^i$ and $\mathbf{u}_2^i$, we can compute the values of the dot products and cost functions.

i	$\mathbf{w}_1 \bullet \mathbf{u}_1^i = a$	$\mathbf{w}_2 \bullet \mathbf{u}_1^i = b$	$\mathbf{w}_1 \bullet \mathbf{u}_2^i = A$	$\mathbf{w}_2 \bullet \mathbf{u}_2^i = B$
0	$\sqrt{3}/2 = .866$	0	$1/2 = .500$	1
1	$\sqrt{2}(\sqrt{3}+1)/4 = .966$	$\sqrt{2}(\sqrt{3}-1)/4 = .259$	$\sqrt{2}(\sqrt{3}-1)/4 = .259$	$\sqrt{2}(\sqrt{3}+1)/4 = .966$
2	1	$1/2 = .500$	0	$\sqrt{3}/2 = .866$
3	$\sqrt{2}(\sqrt{3}+1)/4 = .966$	$\sqrt{2}/2 = .707$	$-\sqrt{2}(\sqrt{3}-1)/4 = -.259$	$\sqrt{2}/2 = .707$
4	$\sqrt{3}/2 = .866$	$\sqrt{3}/2 = .866$	$-1/2 = -.500$	$1/2 = .500$
5	$\sqrt{2}/2 = .707$	$\sqrt{2}(\sqrt{3}+1)/4 = .966$	$-\sqrt{2}/2 = -.707$	$\sqrt{2}(\sqrt{3}-1)/4 = .259$
6	$1/2 = .500$	1	$-\sqrt{3}/2 = -.866$	0

Given vectors: $\mathbf{w}_1 = (\sqrt{3}/2, 1/2)^T$; $\mathbf{w}_2 = (0, 1)^T$.

For the calculation of cost function values of C_1 (and C_2), we compute the values of $a^2, b^2, A^2, B^2, |a|, |b|, |A|, |B|$:

i	a^2	b^2	A^2	B^2	$\lvert a\rvert$	$\lvert b\rvert$	$\lvert A\rvert$	$\lvert B\rvert$
0	$3/4 = .750$	0	.250	1	.866	0	.500	1
1	$(\sqrt{3}+2)/4 = .933$	$(-\sqrt{3}+2)/4 = .067$	.067	.933	.966	.259	.259	.966
2	1	$1/4 = .250$	0	.750	1	.500	0	.866
3	$(\sqrt{3}+2)/4 = .933$	$1/2 = .500$	.067	.500	.966	.707	.259	.707
4	$3/4 = .750$	$3/4 = .750$	.250	.250	.866	.866	.500	.500
5	$1/2 = .500$	$(\sqrt{3}+2)/4 = .933$	.500	.067	.707	.966	.707	.259
6	$1/4 = .250$	1	.750	0	.500	1	.866	0

...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... We can now compute the values of the suggested cost function C_1 (and C_2). Again, they are defined as $C_1 = (a^2-b^2)^2 + (B^2-A^2)^2$ and $C_2 = ||a|-|b|| + ||B|-|A||$, respectively. The following table lists the resulting values:

	i	$(a^2-b^2)^2$	$(B^2-A^2)^2$	$\|\|a\|-\|b\|\|$	$\|\|B\|-\|A\|\|$	C_1	C_2
	0	.563	.563	.866	.500	1.126	1.366
BEST	1	.750	.750	.707	.707	1.5	1.414
	2	.563	.563	.500	.866	1.126	1.366
	3	.187	.187	.259	.448	.374	.707
WORST	4	0	0	0	0	0	0
	5	.187	.187	-.259	-.448	.374	.707
	6	.563	.563	-.500	-.866	1.126	1.366

The figure on page 14, bottom-left (10/8/2023) calls the optimally rotated orthonormal vector pair $\{u_1^{OPT}, u_2^{OPT}\}$.

The table provided on this page shows that the values of cost functions C_1 and C_2 are maximal for the rotation angle $\alpha = \pi/12$ ($i=1$) and minimal for $\alpha = \pi/3$ ($i=4$) — for the finite set of rotation angles used in this numerical example. In fact, for this set of given data an optimal orthonormal vector pair is indeed the pair generated by the considered finite, discrete example, i.e., $\{u_1^{OPT} = (\sqrt{2}(\sqrt{3}+1)/4, \sqrt{2}(\sqrt{3}-1)/4)^T, u_2^{OPT} = (-\sqrt{2}(\sqrt{3}-1)/4, \sqrt{2}(\sqrt{3}+1)/4)^T\}$ — which is clearly implied by the geometry of the given vectors, see bottom-left figure on page 14 (10/8/2023). ...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... Of course, one is interested in the exact analytical definition of a cost function and the location(s) in its domain where the function is extremal. For this purpose, we consider the analytical definition of $C_1 = C_1(a,b,A,B) = (a^2-b^2)^2 + (B^2-A^2)^2$. As mentioned before, we can define the orthonormal vector pair as a pair represent in just one variable (x) by setting:

$\mathbf{u}_1 = (x, \sqrt{1-x^2})^T$, $\mathbf{u}_2 = (-\sqrt{1-x^2}, x)^T$, $0 \le x \le 1$.

(Thus, we only consider points and their positional vectors on the "upper semi-circle," i.e., $x^2+y^2-1=0 \wedge y \ge 0 \wedge -1 \le x \le 1$.) Again, the given unit non-orthogonal vectors are $\mathbf{v}_1 = (x_1, y_1)^T$ and $\mathbf{v}_2 = (x_2, y_2)^T$. For simplicity, we use these abbreviated notations: $\sqrt{\ } = \sqrt{1-x^2}$, $\sqrt{\ }_1 = \sqrt{1-x_1^2}$, $\sqrt{\ }_2 = \sqrt{1-x_2^2}$. We now derive C_1:

$$
\begin{aligned}
C_1 &= (a^2-b^2)^2 + (B^2-A^2)^2 \\
&= \left((\mathbf{u}_1 \cdot \mathbf{v}_1)^2 - (\mathbf{u}_1 \cdot \mathbf{v}_2)^2 \right)^2 \\
&\qquad + \left((\mathbf{u}_2 \cdot \mathbf{v}_2)^2 - (\mathbf{u}_2 \cdot \mathbf{v}_1)^2 \right)^2 \\
&= \left((x_1 x + y_1 \sqrt{\ })^2 - (x_2 x + y_2 \sqrt{\ })^2 \right)^2 \\
&\qquad + \left((-x_2 \sqrt{\ } + y_2 x)^2 - (-x_1 \sqrt{\ } + y_1 x)^2 \right)^2 \\
&= \left((x_1 x)^2 + 2 x_1 y_1 x \sqrt{\ } + y_1^2 (1-x^2) - (x_2 x)^2 - 2 x_2 y_2 x \sqrt{\ } - y_2^2 (1-x^2) \right)^2 \\
&\quad + \left(x_2^2 (1-x^2) - 2 x_2 y_2 x \sqrt{\ } + (y_2 x)^2 - x_1^2 (1-x^2) + 2 x_1 y_1 x \sqrt{\ } - (y_1 x)^2 \right)^2 \\
&= \ldots
\end{aligned}
$$

• • •

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

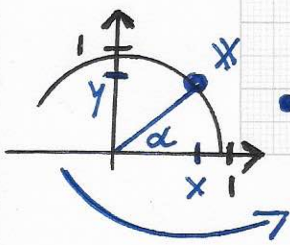
- **Laplacian eigenfunctions and neural networks:...**

$$= \left((x_1^2 - y_1^2 - x_2^2 + y_2^2)\, x^2 + 2(x_1 y_1 - x_2 y_2)\, x\sqrt{\ } + (y_1^2 - y_2^2) \right)^2$$
$$+ \left((-x_2^2 + y_2^2 + x_1^2 - y_1^2)\, x^2 + 2(x_1 y_1 - x_2 y_2)\, x\sqrt{\ } + (x_2^2 - x_1^2) \right)^2$$

{ **Note:**
- $x_1^2 - y_1^2 - x_2^2 + y_2^2 = x_1^2 - (1 - x_1^2) - x_2^2 + (1 - x_2^2) = 2(x_1^2 - x_2^2)$
- $y_1^2 - y_2^2 = (1 - x_1^2) - (1 - x_2^2) = (x_2^2 - x_1^2)$
- $x_1 y_1 - x_2 y_2 = x_1 \sqrt{_1} - x_2 \sqrt{_2}$ }

$$= \left(2(x_1^2 - x_2^2)x^2 + 2(x_1\sqrt{_1} - x_2\sqrt{_2})\, x\sqrt{\ } + (x_2^2 - x_1^2) \right)^2$$
$$+ \left(2(x_1^2 - x_2^2)x^2 + 2(x_1\sqrt{_1} - x_2\sqrt{_2})\, x\sqrt{\ } + (x_2^2 - x_1^2) \right)^2$$
$$= 2\left(2(x_1^2 - x_2^2)x^2 + 2(x_1\sqrt{_1} - x_2\sqrt{_2})\, x\sqrt{\ } + (x_2^2 - x_1^2) \right)^2$$
$$= 2\left(2(x_1^2 - x_2^2)x^2 - (x_1^2 - x_2^2) + 2(x_1\sqrt{_1} - x_2\sqrt{_2})\, x\sqrt{\ } \right)^2$$
$$= 2\left(2\beta x^2 - \beta + 2\gamma x\sqrt{\ } \right)^2$$
$$= 2\left((2\beta x^2 - \beta)^2 + 4(2\beta x^2 - \beta)\gamma x\sqrt{\ } + 4\gamma^2 x^2 (1 - x^2) \right)$$
$$= 2\left(4\beta^2 x^4 - 4\beta^2 x^2 + \beta^2 + 8\beta\gamma x^3\sqrt{\ } - 4\beta\gamma x\sqrt{\ } + 4\gamma^2 x^2 - 4\gamma^2 x^4 \right)$$
$$= 2\left(4(\beta^2 - \gamma^2)x^4 - 4(\beta^2 - \gamma^2)x^2 + 8\beta\gamma x^3\sqrt{\ } - 4\beta\gamma x\sqrt{\ } + \beta^2 \right)$$
$$= 8(\beta^2 - \gamma^2)x^4 - 8(\beta^2 - \gamma^2)x^2 + 8\beta\gamma\, x\sqrt{\ }\,(2x^2 - 1) + 2\beta^2$$
$$= 8(\beta^2 - \gamma^2)(x^2 - 1)x^2 + 8\beta\gamma(2x^2 - 1)\, x\sqrt{\ } + 2\beta^2 = \ldots$$

! This cost function is TOO COMPLICATED for practical, computational use – especially when considering its generalization to the **D**-dimensional case. Nevertheless, this definition of C_1 is correct.



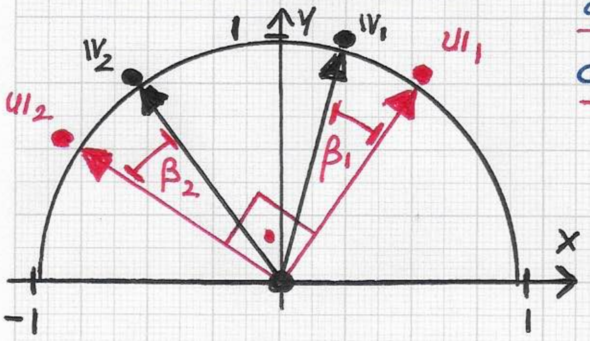
- **Note.** One might be able to simplify this cost function C_1 by using the following: $x = \cos\alpha$, $y = \sin\alpha$; $\sqrt{\ } = y$; $x\sqrt{\ } = \cos\alpha \cdot \sin\alpha$; 2-by-2 determinants defining areas; theorems for trigonometric functions etc. ...

Bernd Hamann 10/13/2023

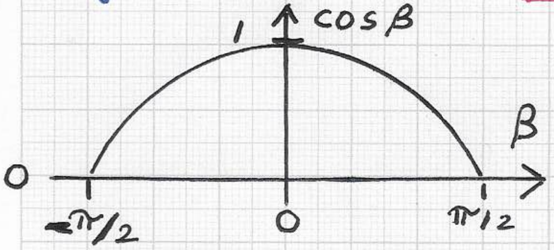
Stratovan

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

- **Laplacian eigenfunctions and neural networks:...**



We now consider a "simpler and direct" approach for the calculation of $\mathbb{u}_1$ and $\mathbb{u}_2$ via an angle requirement. The left figure illustrates all data required. The non-orthogonal vector pair $\{\mathbb{v}_1, \mathbb{v}_2\}$ is given, and we want to compute the/an orthogonal vector pair $\{\mathbb{u}_1, \mathbb{u}_2\}$ such that $\beta_1 = \beta_2$.



Symmetry: $\cos\beta = \cos(-\beta)$.

Given are the unit vectors $\mathbb{v}_1 = (x_1, y_1)^T$, $\mathbb{v}_2 = (x_2, y_2)^T$, and we derive the coordinate values of the vectors $\mathbb{u}_1 = (x, V)^T$, $\mathbb{u}_2 = (-V, x)^T$, where we abbreviate the term $\sqrt{1-x^2}$ again by V. The objective is to determine the directions of $\mathbb{u}_1$ and $\mathbb{u}_2$ such that $\beta_1 = \beta_2$. We perform calculations with $\cos\beta_1$ and $\cos\beta_2$, as the needed cos-values can be obtained by the corresponding dot products of two vectors. In this context, one must therefore keep the symmetry behavior of the cos function in mind: $\cos\beta = \cos(-\beta)$, i.e., β and its negated version have the same cos-value, which is sketched in the second figure on this page.

...