OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

- **Laplacian eigenfunctions and neural networks:...** We can now derive the formula defining u_1 and u_2:

$$\cos\beta_1 = u_1 \bullet w_1 = (x, \surd)^T \bullet (x_1, y_1)^T = x_1 x + y_1 \surd\,;$$

$$\cos\beta_2 = u_2 \bullet w_2 = (-\surd, x)^T \bullet (x_2, y_2)^T = -x_2 \surd + y_2 x\,.$$

$\Rightarrow$ condition: $\cos\beta_1 = \cos\beta_2$

$$\Rightarrow x_1 x + y_1 \surd = -x_2 \surd + y_2 x$$

$$\Rightarrow (x_1 - y_2)\, x = (-x_2 - y_1)\, \surd$$

$$(y_2 - x_1)\, x = (y_1 + x_2)\, \surd \qquad [\text{Eq.}]$$

$\boxed{w_1 \bullet w_2 \neq 0}$

$$\Rightarrow (y_2 - x_1)^2 x^2 = (y_1 + x_2)^2\,(1 - x^2)$$

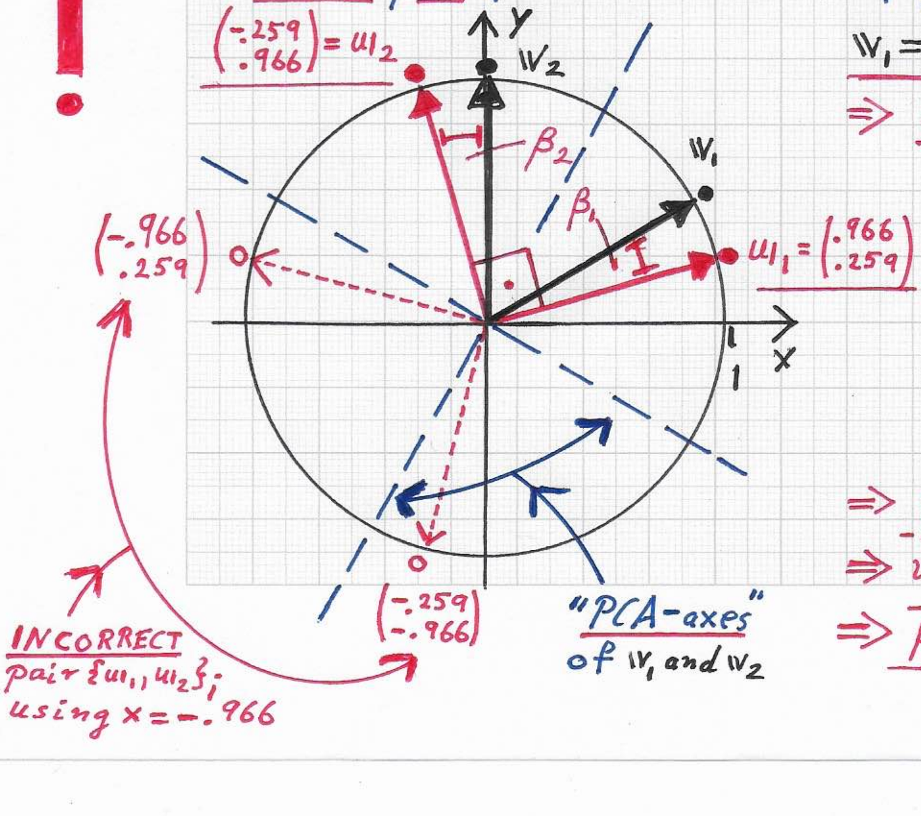
$$\Rightarrow \left((y_1 + x_2)^2 + (y_2 - x_1)^2\right) x^2 = (y_1 + x_2)^2$$

$$\Rightarrow x^2 = (y_1 + x_2)^2 / \left((y_1 + x_2)^2 + (y_2 - x_1)^2\right)$$

$$\Rightarrow x = +/- \sqrt{(y_1 + x_2)^2 / \left((y_1 + x_2)^2 + (y_2 - x_1)^2\right)}$$

$\Rightarrow u_1 = \ldots\,,\; u_2 = \ldots$

- **Example** (see bottom-left figure on page 14 (10/8/2023)):



$w_1 = (\sqrt{3}/2,\ 1/2)^T,\quad w_2 = (0,\ 1)^T$

$$\Rightarrow x = +/- \left[(1/2+0)^2 / \left((1/2+0)^2 + (1-\sqrt{3}/2)^2 \right) \right]^{1/2}$$

$$= +/- \left[1/4 \,/\, (1/4 + 7/4 - \sqrt{3}) \right]^{1/2}$$

$$= +/- \left[1/4 \,/\, (2 - \sqrt{3}) \right]^{1/2}$$

$$= +/- \left[(2+\sqrt{3})/4 \right]^{1/2}$$

$$= +/- \tfrac{1}{2}\sqrt{2+\sqrt{3}} = +/- \left(\sqrt{2}(\sqrt{3}+1)\right)/4$$

$$= +/- \sqrt{2}\,(\sqrt{3}+1)/4 = +/- .966$$

$$\Rightarrow y = \sqrt{1-x^2} = \ldots = \sqrt{2}\,(\sqrt{3}-1)/4 = .259$$

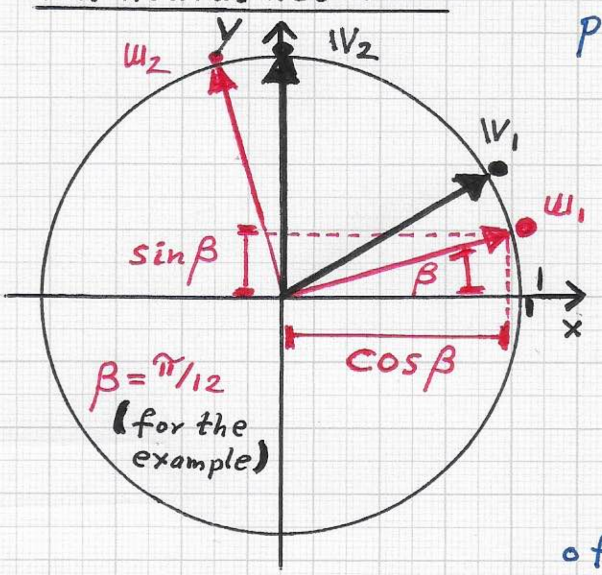
$$\Rightarrow u_1 = (+.966, .259)^T,\ u_2 = (-.259, +.966)^T$$

$$\Rightarrow \beta_1 = \beta_2 = \pi/12 \ (\text{test})$$

Stratovan

OBJECT AND MATERIAL EIGENFUNCTIONS - Cont'd.

• Laplacian eigenfunctions and neural networks:...



• Note. The numerical example provided on the previous page shows that the proposed "simpler and direct" approach can indeed be used to determine optimal values for w_1 and w_2. Unfortunately, the algebraic definition of the x-value of w_1 requires one to select the $x = +\sqrt{\ }$ and not the $x = -\sqrt{\ }$ alternative. It is possible to use the tan function to resolve this ambiguity. The top-left figure illustrates the geometrical configuration needed for another option to calculate w_1 – by computing the angle β. The derivation for the x-value included the equation [Eq.], $(y_2 - x_1)x = (y_1 + x_2)\sqrt{\ }$. By recalling that $\sqrt{\ } = \sqrt{1-x^2} = y$ (y-value of w_1), we can derive the value for β as follows:

(Periodicity: $\tan\beta = \tan(\beta + \pi)$)

$$(y_2 - x_1)x = (y_1 + x_2)y \Rightarrow x/y = (y_1 + x_2)/(y_2 - x_1)$$

$$\Rightarrow \tan\beta = \sin\beta/\cos\beta = (y_2 - x_1)/(y_1 + x_2) = y/x$$

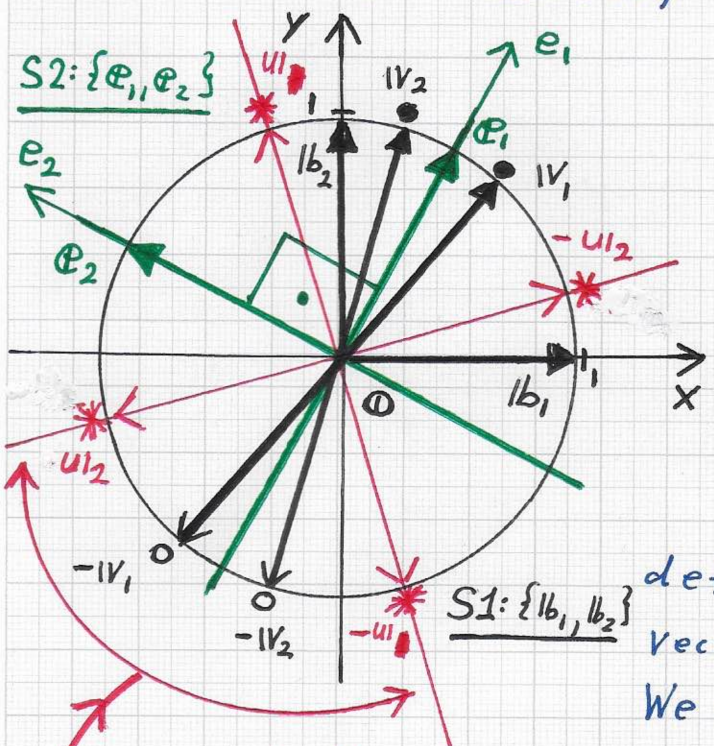
$$\Rightarrow \beta = \arctan\left((y_2 - x_1)/(y_1 + x_2)\right)$$

Thus, for the specific numerical example considered, one obtains $\beta = \arctan((1-\sqrt{3}/2)/(1/2 + 0)) = \arctan(2-\sqrt{3}) = \pi/12 = 15°$. This value is the unique, correct result.

...

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... At this point, we discusse the 2D case from the perspective of the geometry of conic sections and quadrics. (Reference: "Geometric Concepts for Geometric Design" by Wolfgang Boehm and Hartmut Prautzsch, AK Peters, 1994). We consider the general 2D case shown in the left figure. The origin is the point $\mathbb{O}$, and it has unique coordinates $\mathbb{O}=(0,0)^T$. The initial reference coordinate system S1 is defined by the two basis vectors $\mathbb{b}_1=(1,0)^T$ and $\mathbb{b}_2=(0,1)^T$. We are given two unit, non-orthogonal vectors $\mathbb{v}_1=(x_1,y_1)^T$ and $\mathbb{v}_2=(x_2,y_2)^T$. These two vectors together with their negated versions $-\mathbb{v}_1$ and $-\mathbb{v}_2$ define a 2-by-2 covariance matrix C, i.e., $C=\left(\mathbb{v}_1\ \mathbb{v}_2\ -\mathbb{v}_1\ -\mathbb{v}_2\right)\cdot\left(\mathbb{v}_1\ \mathbb{v}_2\ -\mathbb{v}_1\ -\mathbb{v}_2\right)^T=$



perpendicular line pair (degenerate conic section)

$$=\begin{pmatrix} x_1 & x_2 & -x_1 & -x_2 \\ y_1 & y_2 & -y_1 & -y_2\end{pmatrix}\cdot\begin{pmatrix} x_1 & x_2 & -x_1 & -x_2 \\ y_1 & y_2 & -y_1 & -y_2\end{pmatrix}^T=\begin{pmatrix} 2(x_1^2+x_2^2) & 2(x_1y_1+x_2y_2) \\ 2(x_1y_1+x_2y_2) & 2(y_1^2+y_2^2)\end{pmatrix}.$$

The matrix C has two real eigenvalues λ_1 and λ_2 $(|\lambda_1|>|\lambda_2|)$ with associated perpendicular normalized eigenvectors $\mathbb{e}_1$ and $\mathbb{e}_2$; they define the orthonormal system S2. ...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... Thus, the rotation matrix R that maps the basis $\{\mathbb{b}_1, \mathbb{b}_2\}$ of the orthonormal system S1 to the basis $\{\mathbb{e}_1, \mathbb{e}_2\}$ that defines the orthonormal system S2 is given by $R = (\mathbb{e}_1\ \mathbb{e}_2)$, i.e., the matrix R that has $\mathbb{e}_1$ and $\mathbb{e}_2$ as its two columns. We refer to the coordinates of a point represented relative to coordinate system S1 as $(x, y)^T$; the same point represented relative to coordinate system S2 is written as $(e_1, e_2)^T$. Since R is a rotation matrix with normalized orthogonal vectors as its columns, $R^{-1} = R^T$. Thus, the mapping of a point given as a point relative to system S1, i.e., $\mathbb{x}_{S1} = (x, y)^T$, **to** its equivalent representation relative to system S2 is

$$\mathbb{x}_{S2} = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} \leftarrow \mathbb{e}_1^T \rightarrow \\ \leftarrow \mathbb{e}_2^T \rightarrow \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = R^T \cdot \mathbb{x}_{S1} .$$

(The matrix R^T has $\mathbb{e}_1^T$ and $\mathbb{e}_2^T$ as its rows.)

It is now possible to represent a PERPENDICULAR LINE PAIR (passing through the origin $\mathbb{O}$) with respect to system S2 with its e_1- and e_2-axes. Further, we can also represent the unit circle relative to S2. The resulting algebraic definitions are:

(i) $e_1^2 - e_2^2 = 0$ (line pair); (ii) $e_1^2 + e_2^2 = 1$ (circle).

...

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... These definitions for an implicitly defined orthogonal line pair and the implicitly defined unit circle can be used to calculate candidate points/vectors for the needed $\{u_1, u_2\}$ pair by intersecting the lines and the circle:

(i) $e_1^2 - e_2^2 = 0 \Rightarrow e_1^2 = e_2^2$ | insert into (ii)

$\Rightarrow$ (ii) $e_1^2 + e_2^2 = e_1^2 + e_1^2 = 2e_1^2 = 1 \Rightarrow e_1^2 = 1/2$

$$\Rightarrow \quad e_1 = {}^{+}/_{-}\sqrt{2}/2 \;, \quad e_2 = {}^{+}/_{-}\sqrt{2}/2 \,.$$

These four values define the four points/vectors $(\sqrt{2}/2, \sqrt{2}/2)^T$, $(-\sqrt{2}/2, \sqrt{2}/2)^T$, $(-\sqrt{2}/2, -\sqrt{2}/2)^T$ and $(\sqrt{2}/2, -\sqrt{2}/2)^T$. The complex figure on page 23 (10/15/2023) refers to these four points/vectors as u_1, u_2, $-u_1$ and $-u_2$.

THE ORTHOGONAL PAIRS $\{u_1, u_2\}$, $\{u_2, -u_1\}$, $\{-u_1, -u_2\}$ AND $\{-u_2, u_1\}$ DEFINE THE POSSIBLE $2^2 = 2^D = 4$ CANDIDATE PAIRS FROM WHICH ONE CAN SELECT $\{u_1, u_2\}$, FOR EXAMPLE, TO OBTAIN THE DESIRED OPTIMAL ORTHONORMAL BASIS – WHICH IS THE BEST APPROXIMATION TO THE NON-ORTHOGONAL GIVEN PAIR $\{v_1, v_2\}$.

This elegant solution approach, based on the use of **PCA**, the definition of (implicitly defined) conic sections (lines, circle) and the algebraic intersection of conic sections, can be generalized to the D-dimensional case. ...