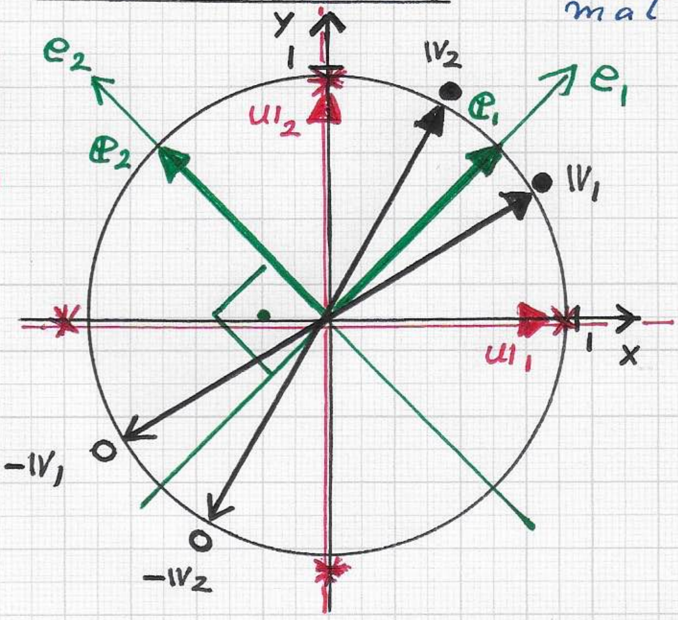
OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:...



Before considering the optimal orthonormal basis construction for the 3D case, we perform the described computations for a numerical 2D example illustrated in the left figure. The given unit non-orthogonal vectors are

$\mathbb{v}_1 = (\sqrt{3}/2, 1/2)^T$, $\mathbb{v}_2 = (1/2, \sqrt{3}/2)^T$.

The covariance matrix defining the eigensystem is

$$\mathbf{C} = (\mathbb{v}_1\ \mathbb{v}_2\ -\mathbb{v}_1\ -\mathbb{v}_2)\cdot(\mathbb{v}_1\ \mathbb{v}_2\ -\mathbb{v}_1\ -\mathbb{v}_2)^T = \begin{pmatrix} x_1 & x_2 & -x_1 & -x_2 \\ y_1 & y_2 & -y_1 & -y_2 \end{pmatrix}\begin{pmatrix} x_1 & x_2 & -x_1 & -x_2 \\ y_1 & y_2 & -y_1 & -y_2 \end{pmatrix}^T$$

$$= \begin{pmatrix} x_1 & x_2 & x_1 & x_2 \\ y_1 & y_2 & y_1 & y_2 \end{pmatrix}\cdot\begin{pmatrix} x_1 & x_2 & x_1 & x_2 \\ y_1 & y_2 & y_1 & y_2 \end{pmatrix}^T = 2\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}\cdot\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}^T$$

$$= 2\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}\cdot\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} = 2\begin{pmatrix} \sum_{i=1}^2 x_i^2 & \sum_{i=1}^2 x_i y_i \\ \sum_{i=1}^2 x_i y_i & \sum_{i=1}^2 y_i^2 \end{pmatrix}$$

$$= 2\begin{pmatrix} 1 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & \sqrt{3} \\ \sqrt{3} & 2 \end{pmatrix}.$$

The two real eigenvalues and corresponding normalized eigenvectors are

$\lambda_1 = 2 + \sqrt{3}$, $\lambda_2 = 2 - \sqrt{3}$; $\mathbb{e}_1 = (\sqrt{2}/2, \sqrt{2}/2)^T$, $\mathbb{e}_2 = (-\sqrt{2}/2, \sqrt{2}/2)^T$.

We refer to the $\mathbb{e}_1$-direction as the dominant direction, since $\lambda_1 > \lambda_2$. Thus, the rotation matrix mapping the original basis vectors $\mathbb{b}_1 = (1,0)^T$ and $\mathbb{b}_2 = (0,1)^T$ is $R = (\mathbb{e}_1\ \mathbb{e}_2) = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}$, and $R^{-1} = \begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}$. ...

■ OBJECT AND MATERIAL EIGENFUNCTIONS - Cont'd.

• Laplacian eigenfunctions and neural networks:... Therefore, the perpendicular line pair defined as the (degenerate) conic section $e_1^2 - e_2^2 = 0$ intersects the unit circle in the four points $(1,0)^T$, $(0,1)^T$, $(-1,0)^T$ and $(0,-1)^T$, shown as '$*$' in the figure on the previous page and represented here relative to system $S1 = \{\mathbb{b}_1, \mathbb{b}_2\}$. With respect to the computed eigensystem $S2 = \{\mathbb{e}_1, \mathbb{e}_2\}$ the four points are the result of algebraically intersecting $e_1^2 - e_2^2 = 0$ and $e_1^2 + e_2^2 = 1$, yielding the points' representation relative to system $S2$, i.e., $(\sqrt{2}/2, -\sqrt{2}/2)^T$, $(\sqrt{2}/2, \sqrt{2}/2)^T$, $(-\sqrt{2}/2, \sqrt{2}/2)^T$ and $(-\sqrt{2}/2, -\sqrt{2}/2)^T$. For example, if one selects the vectors $(\sqrt{2}/2, -\sqrt{2},2)^T$ and $(\sqrt{2}/2, \sqrt{2}/2)^T$ (expressed relative to $S2$), one obtains their corresponding representations relative to system $S1$ as $\mathbb{u}_1 = \mathbb{u}_1^{S1} = R \cdot (\sqrt{2}/2, -\sqrt{2}/2)^T$ and $\mathbb{u}_2 = \mathbb{u}_2^{S1} = R \cdot (\sqrt{2}/2, \sqrt{2}/2)^T$, i.e., $\mathbb{u}_1 = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix} \begin{pmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbb{u}_2 = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 \\ \sqrt{2}/2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

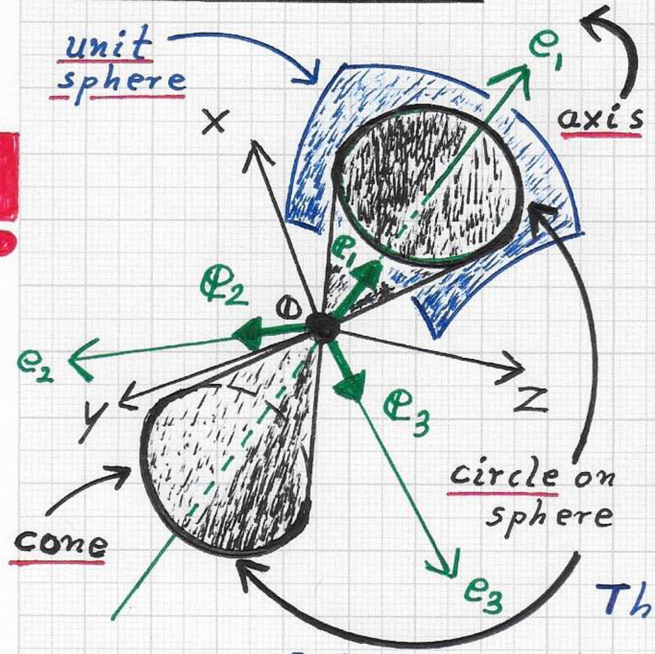
The normalized and orthogonal vectors $\mathbb{u}_1$ and $\mathbb{u}_2$ define an optimal, best approximation of the given vector pair $\{\mathbb{v}_1, \mathbb{v}_2\}$; they are indicated in the figure on the previous page.

• **Note.** The representations of a point/vector relative to systems $S1$ and $S2$ are $\mathbb{p}^{S2} = R^T \mathbb{p}^{S1}$ and $\mathbb{p}^{S1} = R\,\mathbb{p}^{S2}$, with $\mathbb{p}$ denoting the point/vector. • • •

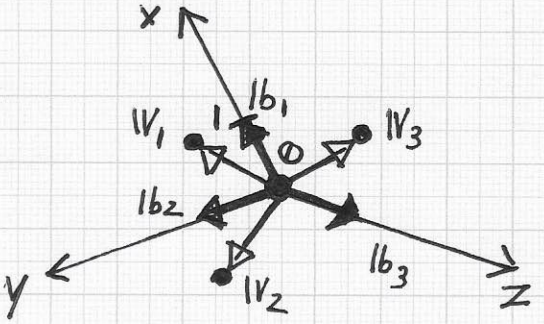
Stratovan

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:...



The generalization of the presented "optimal ortho-normalization" method from the 2-dimensional to the D-dimensional setting is rather involved, and it is thus helpful to consider the generalization to the 3-dimensional case. The left figure illustrates some of the geometrical and algebraic primitives and concepts one must employ in the 3D case.



Initial coordinate system.

In the 3D case, the original ("global") coordinate system is defined by the origin $\mathbb{O}$ and the (right-handed) set of orthonormal basis vectors, $\{\mathbb{b}_1=(1,0,0)^T, \mathbb{b}_2=(0,1,0)^T, \mathbb{b}_3=(0,0,1)^T\}$, see second figure on this page. The given unit non-orthogonal vectors for which an optimal, best orthonormal basis vector set $\{\mathbb{u}_1, \mathbb{u}_2, \mathbb{u}_3\}$ must be constructed are the vectors $\mathbb{V}_i=(x_i, y_i, z_i)^T$, $i=1,2,3$.

The top figure shows eigensystem vectors $\mathbb{e}_i$, $i=1,2,3$, implied by the given vectors $\mathbb{V}_i$, an "implied cone" and this cone's intersection circles on the unit sphere.

...

Stratovan

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks: ...

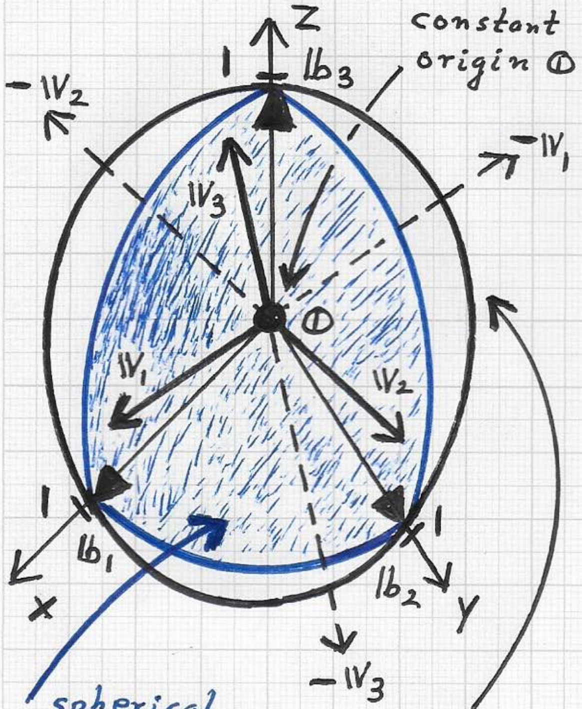
In the 3D case, the construction of the eigensystem defined by its defining normalized eigenvectors $\mathbb{e}_1$, $\mathbb{e}_2$ and $\mathbb{e}_3$ is again implied by the covariance matrix $\mathbf{C}$, induced by the given unit non-orthogonal vectors $\mathbb{v}_1$, $\mathbb{v}_2$ and $\mathbb{v}_3$ — see left figure — as follows:

$$\mathbf{C} = \left(\mathbb{v}_1\ \mathbb{v}_2\ \mathbb{v}_3\ {-\mathbb{v}_1}\ {-\mathbb{v}_2}\ {-\mathbb{v}_3}\right)\cdot\left(\mathbb{v}_1\ \mathbb{v}_2\ \mathbb{v}_3\ {-\mathbb{v}_1}\ {-\mathbb{v}_2}\ {-\mathbb{v}_3}\right)^T,$$

where $\mathbb{v}_i = (x_i, y_i, z_i)^T$, $i = 1, 2, 3$.

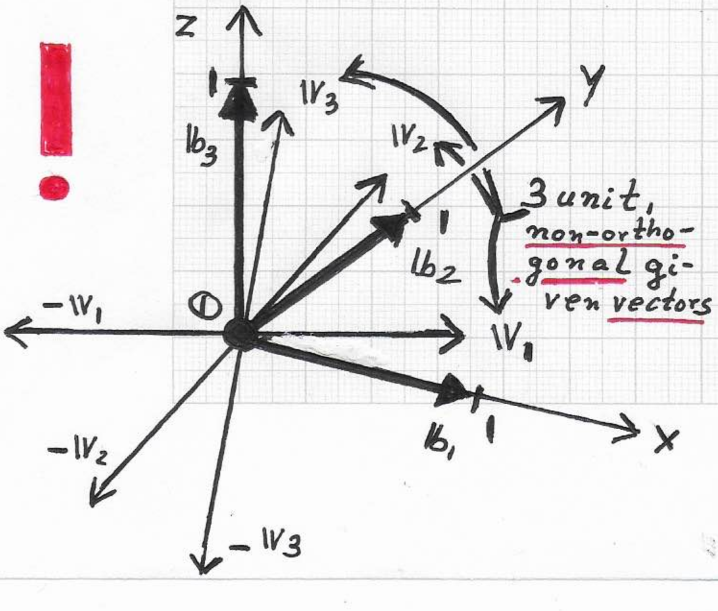
Thus $\mathbf{C}$ is the symmetric matrix

$$\mathbf{C} = 2\cdot\begin{bmatrix} \sum_{i=1}^{3} x_i^2 & \sum_{i=1}^{3} x_i y_i & \sum_{i=1}^{3} x_i z_i \\ \sum_{i=1}^{3} y_i x_i & \sum_{i=1}^{3} y_i^2 & \sum_{i=1}^{3} y_i z_i \\ \sum_{i=1}^{3} z_i x_i & \sum_{i=1}^{3} z_i y_i & \sum_{i=1}^{3} z_i^2 \end{bmatrix}.$$



spherical triangular patch — shown for context — with corners $(1,0,0)^T$, $(0,1,0)^T$ and $(0,0,1)^T$

circle passing through points $(1,0,0)^T$, $(0,1,0)^T$ and $(0,0,1)^T$ — shown for context



We consider the specific example sketched in the left figure, where $\mathbb{v}_1 = \sqrt{2}\cdot(4/6, 1/6, 1/6)^T$, $\mathbb{v}_2 = \sqrt{2}\cdot(1/6, 4/6, 1/6)^T$, $\mathbb{v}_3 = \sqrt{2}\cdot(1/6, 1/6, 4/6)^T$. Vector $\mathbb{v}_i$ is "very close" to vector $\mathbb{b}_i$; the goal is to optimally orthonormalize $\{\mathbb{v}_i\}_{i=1}^{3}$; intuitively, the result should be $\{\mathbb{b}_i\}_{i=1}^{3}$. •••

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

- Laplacian eigenfunctions and neural networks:... (One can verify that $\mathbb{v}_1$, for example, is a unit vector:

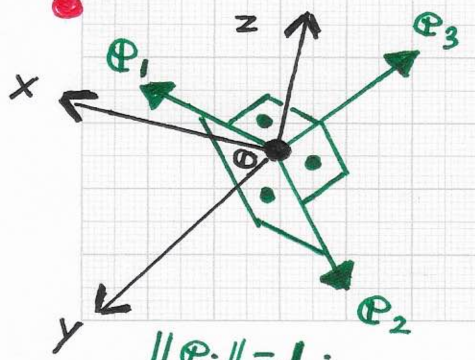
$$\|\mathbb{v}_1\| = \left((2\sqrt{2}/3)^2 + (\sqrt{2}/6)^2 + (\sqrt{2}/6)^2 \right)^{1/2} = \left(8/9 + 1/18 + 1/18 \right)^{1/2} = 1.)$$

For this set of vectors $\mathbb{v}_i$, the resulting covariance matrix is

$$C = 2 \cdot \begin{bmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

The eigenvalues of C are $\lambda_1 = 4$, $\lambda_2 = 1$ and $\lambda_3 = 1$.

The associated normalized eigenvectors are $\mathbb{e}_1 = (\sqrt{3}/3, \sqrt{3}/3, \sqrt{3}/3)^T$, $\mathbb{e}_2 = (-\sqrt{2}/2, \sqrt{2}/2, 0)^T$ and $\mathbb{e}_3 = (-\sqrt{2}/2, 0, \sqrt{2}/2)^T$. The eigenvector associated with the largest eigenvalue will serve as direction vector of the axis of a cone to be defined; here, $\mathbb{e}_1$ will be the direction vector of the cone's axis. **NOTE.** SINCE $\lambda_2 = \lambda_3 = 1$, EIGENVECTORS $\mathbb{e}_2$ and $\mathbb{e}_3$ ARE NOT ORTHOGONAL. WHENEVER NON-ORTHOGONAL EIGENVECTORS RESULT – AS A CONSEQUENCE OF EIGENVALUES WITH MULIPLICITIES LARGER THAN ONE (DEGENERATE CASES) – ONE MUST PERFORM "SIMPLE ORTHOGONALIZATION."



$\|\mathbb{e}_i\| = 1$;
$\mathbb{e}_i \cdot \mathbb{e}_j = 0,\ i \neq j$

In this particular example, one can "re-define" $\mathbb{e}_3$ by using the cross product of $\mathbb{e}_1$ and $\mathbb{e}_2$ (themselves being unit and orthogonal vectors), i.e., $\mathbb{e}_3 := \mathbb{e}_1 \times \mathbb{e}_2 = (-\sqrt{6}/6, -\sqrt{6}/6, \sqrt{6}/3)^T$, see left figure. (The vectors $\mathbb{e}_2$ and $\mathbb{e}_3$ still define the same 2D sub-space, a plane.)

...