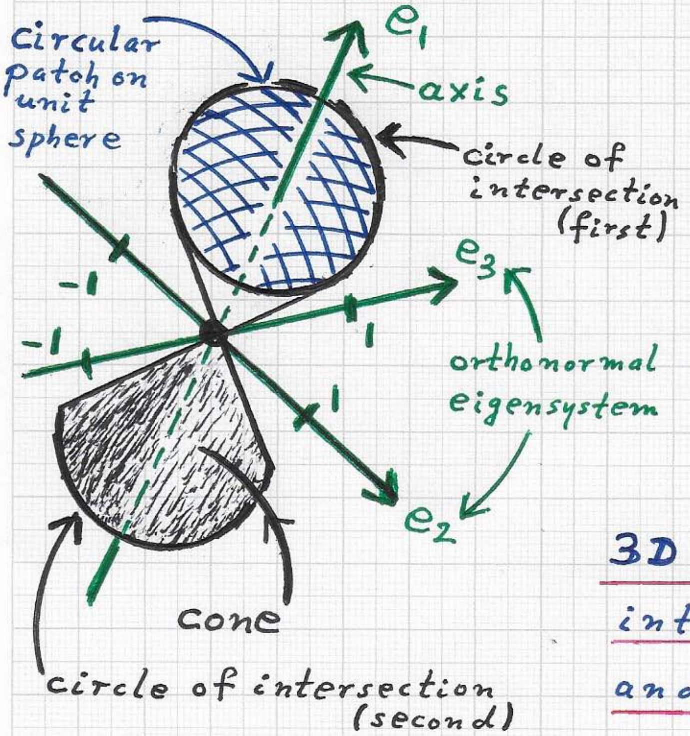
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... The described PCA-based method for the construction of the desired eigensystem "generally" yields eigenvalues that are different, each eigenvalue having multiplicity one; only in singular, degenarate cases eigenvalues with multiplicities larger than one can arise. One can detect such a case and perform the necessary orthonormalization step(s) in a straightforward manner.



On pages 24-25 (10/16/2023), the algebraic definitions for the unit circle and a line pair were provided, used to compute four intersection points. The left figure illustrates the generalization to the 3D case. Here, we compute the intersection of the unit sphere and a (double) cone. We use the constructed eigensystem with its e_1-, e_2- and e_3-axes to represent and intersect the considered geometrical entities. The algebraic definitions are: (i) $-e_1^2+e_2^2+e_3^2=0$ (double cone); (ii) $e_1^2+e_2^2+e_3^2=1$ (unit sphere).

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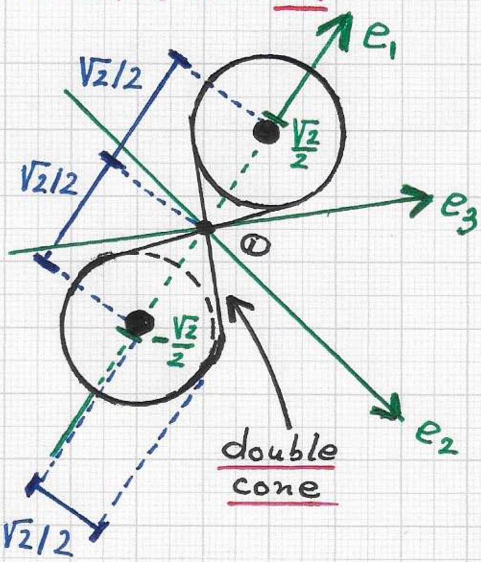
Bernd Hamann
Stratovan

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

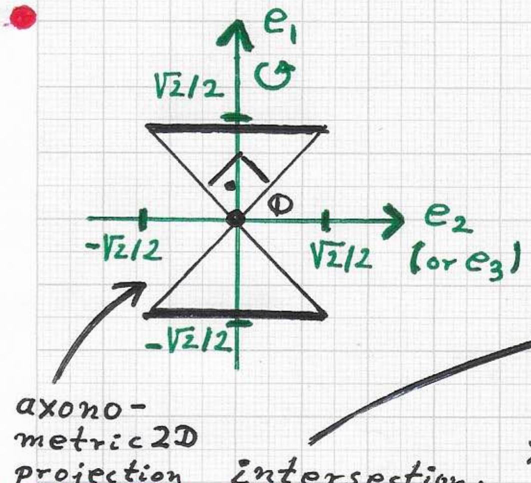
• **Laplacian eigenfunctions and neural networks: ...** Following the approach we employ for the 2D case, we first compute the intersection of the implicitly defined surfaces, the double cone and the unit sphere, by adding their defining algebraic representations (i) and (ii):

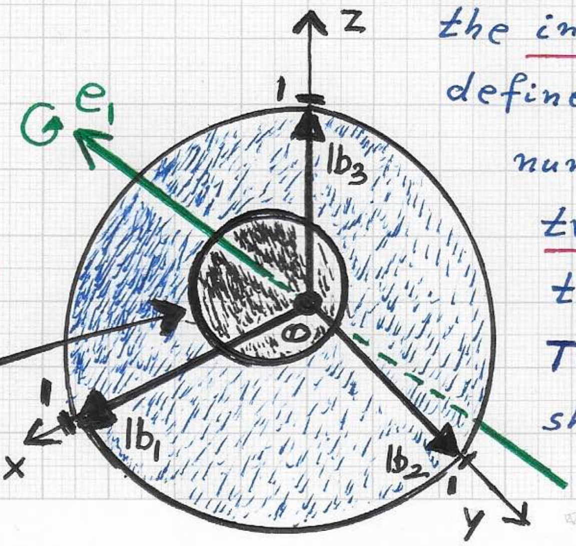
(i) + (ii): $2(e_2^2+e_3^2)=1 \Rightarrow e_2^2+e_3^2 = 1/2 = (\sqrt{2}/2)^2$

insert into (i): $e_1^2 = 1/2 \Rightarrow e_1 = +/- \sqrt{2}/2$



The left figure is an attempt to show the intersection, i.e., two circles $e_2^2+e_3^2=(\sqrt{2}/2)^2$ on the double cone for the two e_1-values $e_1 = +/- \sqrt{2}/2$. The bottom-left figure shows a projection into the e_1-e_2 plane (or e_1-e_3 plane). Effectively, the intersection defines – in this numerical case – two circles on the unit sphere. The left figure shows one of these circles.





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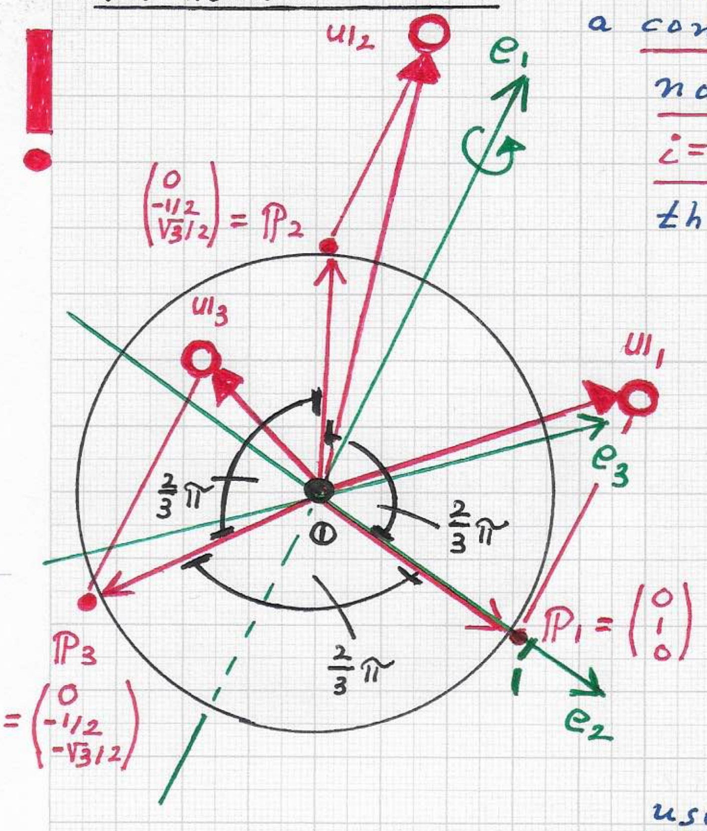
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• **Laplacian eigenfunctions and neural networks:...**



The geometrical approach for a construction of three orthonormal points/vectors $\mathbb{u}_i$, $i=1,2,3$, is illustrated in the left figure — which is not employing a proper perspective drawing technique; it merely portraits the geometrical primitives and steps involved. It is important to point out that the construction uses the eigensystem with its orthonormal basis vectors $\mathbb{e}_1$, $\mathbb{e}_2$ and $\mathbb{e}_3$ and the corresponding e_1-, e_2- and e_3-axis as reference coordinate system. First, we establish three points $\mathbb{p}_1$, $\mathbb{p}_2$ and $\mathbb{p}_3$ in the plane defined by the origin $\mathbb{0}$ and basis vectors $\mathbb{e}_2$ and $\mathbb{e}_3$: We define $\mathbb{p}_1$ by setting $\mathbb{p}_1 = (0,1,0)^T$ and rotating $\mathbb{p}_1$ by $\frac{2}{3}\pi$ and $\frac{4}{3}\pi$ (relative to the e_1-axis as rotation axis). The two 2D rotation matrices for these angles are

$$\begin{bmatrix} \cos\frac{2}{3}\pi & -\sin\frac{2}{3}\pi \\ \sin\frac{2}{3}\pi & \cos\frac{2}{3}\pi \end{bmatrix} = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}; \begin{bmatrix} \cos\frac{4}{3}\pi & -\sin\frac{4}{3}\pi \\ \sin\frac{4}{3}\pi & \cos\frac{4}{3}\pi \end{bmatrix} = \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix}.$$

The top-left figure shows $\mathbb{p}_1$, $\mathbb{p}_2$ and $\mathbb{p}_3$, with coordinates. • • •

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks: ... Second, the points $\mathbb{p}_1$, $\mathbb{p}_2$ and $\mathbb{p}_3$ must be "translated" in e_1-axis direction, yielding points u_1, u_2 and u_3 with positional vectors that are mutually orthogonal, i.e., $u_1 \cdot u_2 = u_1 \cdot u_3 = u_2 \cdot u_3 = 0$. Relative to the eigensystem, these three positional vectors are: $u_1 = (e_1, 1, 0)^T$, $u_2 = (e_1, -1/2, \sqrt{3}/2)^T$ and $u_3 = (e_1, -1/2, -\sqrt{3}/2)^T$. The three orthogonality requirements are:

$$\begin{aligned}
&(i) && u_1 \cdot u_2 = e_1^2 - 1/2 = 0 && \Rightarrow e_1 = +/- \sqrt{2}/2 \\
&(ii) && u_1 \cdot u_3 = e_1^2 - 1/2 = 0 && \Rightarrow e_1 = +/- \sqrt{2}/2 \\
&(iii) && u_2 \cdot u_3 = e_1^2 + 1/4 - 3/4 = 0 && \Rightarrow e_1 = +/- \sqrt{2}/2
\end{aligned}$$

Thus, the desired three orthogonal positional vectors (with positive first coordinate) are: $u_1 = (\sqrt{2}/2, 1, 0)^T$, $u_2 = (\sqrt{2}/2, -1/2, \sqrt{3}/2)^T$ and $u_3 = (\sqrt{2}/2, -1/2, -\sqrt{3}/2)^T$. Further, we compute the length of these vectors and perform normalization: $\|u_1\| = \|u_2\| = \|u_3\| = \sqrt{6}/2$; dividing the coordinate values of u_1, u_2 and u_3 by $\sqrt{6}/2$ yields the normalized versions of these vectors: $u_1 = (\sqrt{3}/3, \sqrt{6}/3, 0)^T$, $u_2 = (\sqrt{3}/3, -\sqrt{6}/6, \sqrt{2}/2)^T$ and $u_3 = (\sqrt{3}/3, -\sqrt{6}/6, -\sqrt{2}/2)^T$ Here, these vectors are represented relative to the eigensystem; as a consequence the average of these vectors is $(u_1 + u_2 + u_3)/3 = (\sqrt{3}/3, 0, 0)^T$. At this point, it is important to understand the geometrical relationship between the vector sets / bases $\{\mathbb{b}_i\}$, $\{\mathbb{v}_i\}$, $\{e_i\}$, and $\{u_i\}$, $i = 1, 2, 3$, to realize these bases' roles in the overall optimization process.

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■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... It will be helpful to summarize the vectors and their defined or derived coordinate values in a table. We must keep in mind that all coordinate systems associated with these basis vector sets share the same global, constant origin $\mathbb{O}$. For simplicity, we write all vectors as rows in the following table. The table provides the exact and approximate values.

$\mathbb{b}_1$	$\mathbb{b}_2$	$\mathbb{b}_3$
$(1, 0, 0)$	$(0, 1, 0)$	$(0, 0, 1)$
$\mathbb{v}_1$	$\mathbb{v}_2$	$\mathbb{v}_3$
$(2\sqrt{2}/3, \sqrt{2}/6, \sqrt{2}/6)$ $= (.94, .24, .24)$	$(\sqrt{2}/6, 2\sqrt{2}/3, \sqrt{2}/6)$ $= (.24, .94, .24)$	$(\sqrt{2}/6, \sqrt{2}/6, 2\sqrt{2}/3)$ $= (.24, .24, .94)$
$\mathbb{e}_1$	$\mathbb{e}_2$	$\mathbb{e}_3$
$(\sqrt{3}/3, \sqrt{3}/3, \sqrt{3}/3)$ $= (.58, .58 .58)$	$(-\sqrt{2}/2, \sqrt{2}/2, 0)$ $= (-.71, .71, 0)$	$(-\sqrt{6}/6, -\sqrt{6}/6, \sqrt{6}/3)$ $= (-.41, -.41, .82)$
(*) $\mathbb{u}_1$	$\mathbb{u}_2$	$\mathbb{u}_3$
$(\sqrt{3}/3, \sqrt{6}/3, 0)$ $= (.58, .82, 0)$	$(\sqrt{3}/3, -\sqrt{6}/6, \sqrt{2}/2)$ $= (.58, -.41, .71)$	$(\sqrt{3}/3, -\sqrt{6}/6, -\sqrt{2}/2)$ $= (.58, -.41, -.71)$

(*) The coordinates provided for these $\mathbb{u}$-vectors represent these vectors with respect to the eigensystem vectors $\mathbb{e}_1$, $\mathbb{e}_2$ and $\mathbb{e}_3$; these vectors $\mathbb{e}_1$, $\mathbb{e}_2$ and $\mathbb{e}_3$ themselves are represented relative to basis vectors $\mathbb{b}_1$, $\mathbb{b}_2$ and $\mathbb{b}_3$.

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