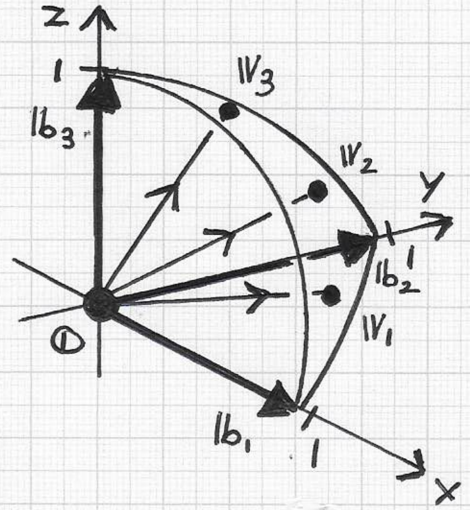
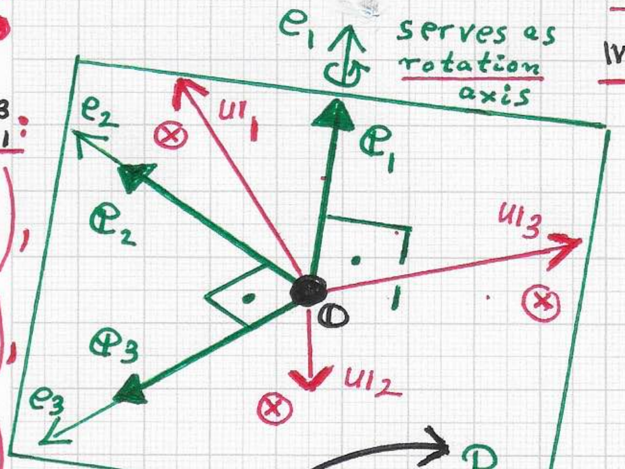
Stratovan

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• **Laplacian eigenfunctions and neural networks:...** Simple drawings of these related vectors are helpful for understanding the construction of the desired optimal, best orthonormal vector approximation. The figures on this page attempt to illustrate the intricate geometry in 3D space via simple projections. The first figure shows the global, "world" coordinate system with origin $\mathbb{O}$ and orthonormal basis vectors $\mathbb{b}_1$, $\mathbb{b}_2$ and $\mathbb{b}_3$ — and the given unit, non-orthogonal vectors $\mathbb{v}_1$, $\mathbb{v}_2$ and $\mathbb{v}_3$. The second figure shows the eigensystem vectors $\mathbb{e}_1$, $\mathbb{e}_2$ and $\mathbb{e}_3$, where the unit vector $\mathbb{e}_1$ is the upward normal vector of plane $\mathbb{P}$, and $\mathbb{P}$ contains the orthonormal vectors $\mathbb{e}_2$ and $\mathbb{e}_3$. The orthonormal vectors $\mathbb{u}_1$, $\mathbb{u}_2$ and $\mathbb{u}_3$ serve as the initial set of vectors to be used for the BEST APPROXIMATION OF $\mathbb{v}_1$, $\mathbb{v}_2$ and $\mathbb{v}_3$ VIA AN OPTIMAL ROTATION OF $\mathbb{u}_1$, $\mathbb{u}_2$ AND $\mathbb{u}_3$ AROUND THE $\mathbb{e}_1$-AXIS. ...





⊗ Relative to basis $\{\mathbb{b}_i\}_{i=1}^{3}$:

$$\mathbb{u}_1 = \begin{pmatrix} (1-\sqrt{3})/3 \\ (1+\sqrt{3})/3 \\ 1/3 \end{pmatrix}, \quad \mathbb{u}_2 = \begin{pmatrix} 1/3 \\ (1-\sqrt{3})/3 \\ (1+\sqrt{3})/3 \end{pmatrix}, \quad \mathbb{u}_3 = \begin{pmatrix} (1+\sqrt{3})/3 \\ 1/3 \\ (1-\sqrt{3})/3 \end{pmatrix}.$$

$\Rightarrow \mathbb{u}_i \cdot \mathbb{u}_j = 0,\ i \neq j,\ \|\mathbb{u}_i\| = 1$

– plane $\mathbb{P}$:

$$x \cdot \sqrt{3}/3 + y \cdot \sqrt{3}/3 + z \cdot \sqrt{3}/3 = 0$$

(eigenvector $\mathbb{e}_1$ is unit plane normal)

OBJECT AND MATERIAL EIGENFUNCTIONS — Cont'd.

• Laplacian eigenfunctions and neural networks:... • **Note.** Before considering the calculation of a best orthonormal vector approximatia via an optimal rotation in 3D space, we briefly discuss two simple related optimization problems. The presented solutions to these problems can be employed for solving the 3D space rotation problem.

(i) Optimal approximation of points on one line/axis:

The left figure sketches the best approximation problem. The values x_1, x_2 and x_3 are given. They must be optimally approximated by an equi-distantly spaced value triple x, $x+1$ and $x+2$. Thus, one must determine an optimal value of x, such that the given x_i-values are optimally approximated by the uniformly spaced x-triple. First, one must establish an appropriate error function to be minimized. For example, one can define a quadratic error function as follows:

$$\begin{aligned}
E(x) = {} & (x-x_1)^2 + (x-x_2)^2 + (x-x_3)^2 + \\
& ((x+1)-x_1)^2 + ((x+1)-x_2)^2 + ((x+1)-x_3)^2 + \\
& ((x+2)-x_1)^2 + ((x+2)-x_2)^2 + ((x+2)-x_3)^2 \\
= {} & \ldots = 9x^2 + 6\,(3-(x_1+x_2+x_3))\,x + 3\big((x_1^2+x_2^2+x_3^2) \\
& \qquad -2(x_1+x_2+x_3)+5\big) \\
= {} & 9x^2 + 6\,(3-(x_1+x_2+x_3))\,x + \underbrace{C}_{\text{constant}}
\end{aligned}$$

OBJECT AND MATERIAL EIGENFUNCTIONS - Cont'd.

- **Laplacian eigenfunctions and neural networks:...** As we want to minimize $E(x)$, we must determine where $d/dx\, E = 0$:

$$\frac{d}{dx} E(x) = 18x + 6(3 - (x_1 + x_2 + x_3)) = 0$$

$$\Rightarrow \quad 3x = (x_1 + x_2 + x_3) - 3$$

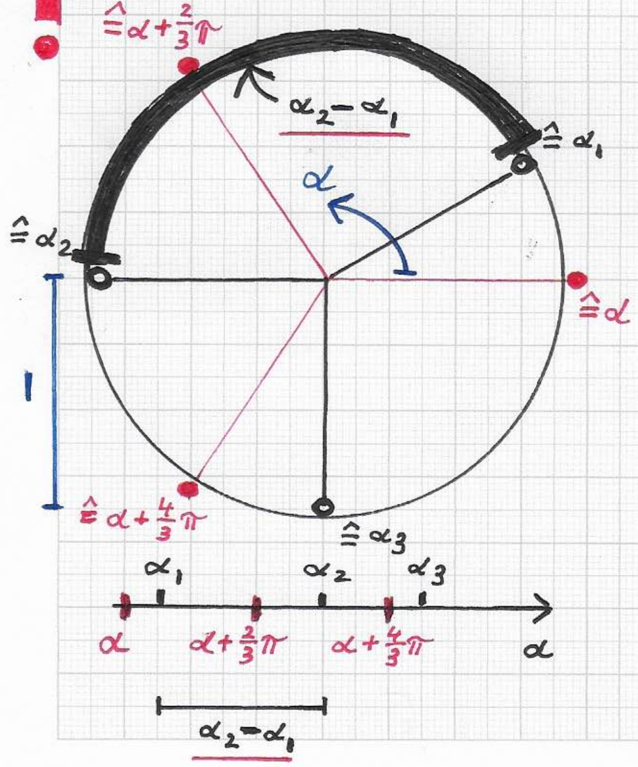
$$\Rightarrow \quad x = -1 + (x_1 + x_2 + x_3)/3 = x_{opt}$$

(Since $d^2/dx^2\, E(x) = 18 > 0$, x_{opt} defines a minimum.)

Thus, this specific error function yields a minimal value when "the value of x is the average of x_1, x_2 and x_3 minus one."

Example: $x_1 = 1$, $x_2 = 2$, $x_3 = 6 \Rightarrow x = 2$.

(ii) Optimal approximation of points on a circle:



The left figure shows the best approximation problem for the unit circle. Three points 'o' are given on the circle. They must be optimally approximated by three equidistantly spaced points on the unit circle. Lengths/distances on the unit circle use arc length. The figure points out that the arc length of the arc between two points is equal to the angle between them — $\alpha_2 - \alpha_1$.

...

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... Since the unit circle "has no beginning and no end," we do not designate a "start point" of the circle; as far as the unit circle's α-parametrization is concerned, one only must keep the period of 2π in mind. Thus, we can use an equidistantly spaced parameter triple, with the spacing of $\frac{2}{3}\pi$ between them – since the goal is the optimal placement of three points on the circle. The parameter triple is α, $\alpha + \frac{2}{3}\pi$ and $\alpha + \frac{4}{3}\pi$. Therefore, one must determine an optimal value of α, such that the given points 'o' on the unit circle are optimally approximated by three points on the unit circle – where the arc length distance between two neighbor points is $\frac{2}{3}\pi$. Formally, we can employ the same error function that defines the approximation error used for case (i). The quadratic error function becomes:

$$
\begin{aligned}
E(\alpha) = {} & (\alpha-\alpha_1)^2 + (\alpha-\alpha_2)^2 + (\alpha-\alpha_3)^2 + \\
& ((\alpha+\tfrac{2}{3}\pi)-\alpha_1)^2 + ((\alpha+\tfrac{2}{3}\pi)-\alpha_2)^2 + ((\alpha+\tfrac{2}{3}\pi)-\alpha_3)^2 + \\
& ((\alpha+\tfrac{4}{3}\pi)-\alpha_1)^2 + ((\alpha+\tfrac{4}{3}\pi)-\alpha_2)^2 + ((\alpha+\tfrac{4}{3}\pi)-\alpha_3)^2 \\
= {} & \ldots = 9\alpha^2 + 6(2\pi - (\alpha_1+\alpha_2+\alpha_3))\alpha + C
\end{aligned}
$$

constant

Thus, $E(\alpha)$ has an extremal value when $d/d\alpha\, E(\alpha) = 0$:

$$d/d\alpha\, E(\alpha) = 18\alpha + 6(2\pi - (\alpha_1+\alpha_2+\alpha_3)) = 0$$

...

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

- Laplacian eigenfunctions and neural networks:... $\Rightarrow 18\alpha = 6((\alpha_1+\alpha_2+\alpha_3)-2\pi)$

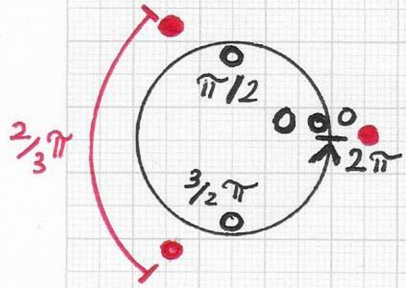
$$\Rightarrow 3\alpha = (\alpha_1+\alpha_2+\alpha_3)-2\pi$$

$$\Rightarrow \alpha = -\tfrac{2}{3}\pi + (\alpha_1+\alpha_2+\alpha_3)/3 = \alpha_{opt}$$

(This α-value represents an extremal value of $E(\alpha)$, and this value – α_{opt} – defines a minimum of $E(\alpha)$, since $\frac{d^2}{d\alpha^2}E(\alpha) = 18 > 0$.)

At this point, it is necessary to know/establish the specific values of α_1, α_2 and α_3, relative to an appropriate reference system, for the computation of α and the subsequently implied optimally placed points '●'.

We consider a simple example. Since the described optimization scheme is not affected by the indices of given points 'o', we do not consider indices in the example sketched in the left figure. Further, we identify a given point 'o' by its parameter value – shown in the figure next to a point 'o', where the unit circle starts ($\alpha = 0$) and ends ($\alpha = 2\pi$) at the circle's right most point with the circle oriented counter-clockwise.



Using the formula for α_{opt}, we obtain $\alpha = -\tfrac{2}{3}\pi + (0 + \tfrac{1}{2}\pi + \tfrac{3}{2}\pi)/3 = -\tfrac{2}{3}\pi + (2\pi)/3 = 0$. Thus, we obtain the equidistantly spaced optimal point triple '●' shown in the figure, with associated parameter values $0, \tfrac{2}{3}\pi, \tfrac{4}{3}\pi$.

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