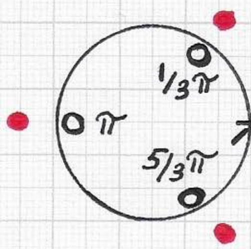
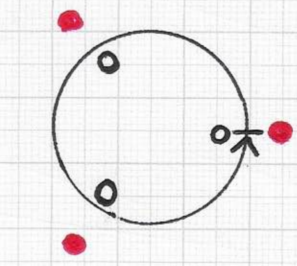
OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• **Laplacian eigenfunctions and neural networks:...** We consider two additional examples for the optimal computation of $\alpha = \alpha_{opt}$ for two simple sets of the given points 'o'. We use the same parameter value arrangements and used in and sketched for the first example discussed on the previous page.

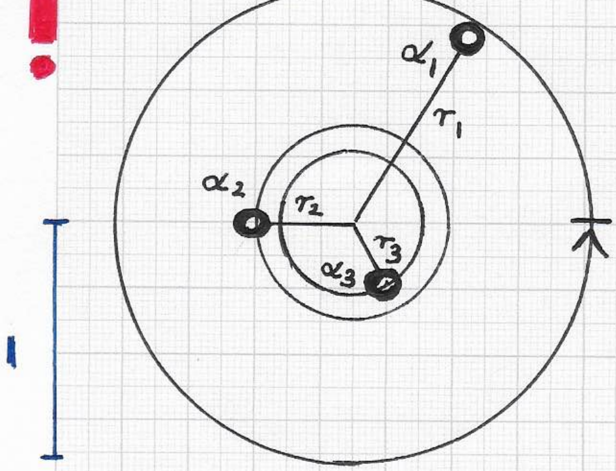




These two additional examples are shown in the two left figures. Both given sets of points 'o' are already optimal. Thus, the proposed optimization method should reproduce the given data exactly. The first set is defined by the parameter triple $\frac{1}{3}\pi, \pi, \frac{5}{3}\pi$. We obtain $\alpha_{opt} = \alpha = -\frac{2}{3}\pi + (\frac{1}{3}\pi + \pi + \frac{5}{3}\pi)/3 = -\frac{2}{3}\pi + (3\pi)/3 = \frac{1}{3}\pi$. The optimal parameter value triple is $\frac{1}{3}\pi, \pi, \frac{5}{3}\pi$ – as expected. The next example uses as given points the points 'o' with parameter values $0\pi, \frac{2}{3}\pi, \frac{4}{3}\pi$. The resulting optimal parameter value is $\alpha_{opt} = \alpha = -\frac{2}{3}\pi + (0\pi + \frac{2}{3}\pi + \frac{4}{3}\pi)/3 = -\frac{2}{3}\pi + (2\pi)/3 = 0\pi$. Consequently, the optimal parameter triple is $0\pi, \frac{2}{3}\pi, \frac{4}{3}\pi$ – as expected, again. The proposed optimization approach and the examples considered implicitly use the ASSUMPTION THAT THE GIVEN POINTS 'o' LIE EXACTLY ON THE UNIT CIRCLE. When one cannot make this assumption, one must generalize the error function. ...

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

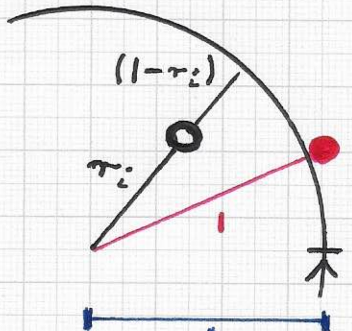
• Laplacian eigenfunctions and neural networks:...



Given points 'o' on circles with radii r_1, r_2, r_3.

It is not obvious how one should generalize the proposed optimization approach and error function to be minimized for the more general case sketched in the left figure: Three points 'o' are given, one lies on the unit circle but two points lie on two different circles with smaller radii (but having the same center).

In such a more general setting, where should one place three equidistantly spaced – in terms of arc length distance – points '•' on the unit circle to optimally approximate the points 'o'? A simple and straightforward answer to this question is this one:



Define weight for point 'o' based on radius r_i, e.g., use $w(r_i) = 1 + (1-r_i)^2$.

Use the "radius r_i" of a point 'o' to define a weight $w(r_i)$ and multiply the corresponding terms of the error function $E(\alpha)$ by $w(r_1), w(r_2), w(r_3)$:

$$
\begin{aligned}
E(\alpha) = {} & w(r_1)\{(\alpha-\alpha_1)^2 + (\alpha+2/3\pi-\alpha_1)^2 + (\alpha+4/3\pi-\alpha_1)^2\} \\
& + w(r_2)\{(\alpha-\alpha_2)^2 + (\alpha+2/3\pi-\alpha_2)^2 + (\alpha+4/3\pi-\alpha_2)^2\} \\
& + w(r_3)\{(\alpha-\alpha_3)^2 + (\alpha+2/3\pi-\alpha_3)^2 + (\alpha+4/3\pi-\alpha_3)^2\} \\
= {} & \ldots
\end{aligned}
$$

...

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• **Laplacian eigenfunctions and neural networks:...**

$$= w(r_1)\{\alpha^2 - 2\alpha_1\alpha + \alpha_1^2 + (\alpha + 2/3\pi)^2 - 2\alpha_1(\alpha + 2/3\pi) + \alpha_1^2 + (\alpha + 4/3\pi)^2 - 2\alpha_1(\alpha + 4/3\pi) + \alpha_1^2$$

$$+ w(r_2)\{\ldots\} + w(r_3)\{\alpha^2 - 2\alpha_3\alpha + \alpha_3^2 + (\alpha + 2/3\pi)^2 - 2\alpha_3(\alpha + 2/3\pi) + \alpha_3^2 + (\alpha + 4/3\pi)^2 - 2\alpha_3(\alpha + 4/3\pi) + \alpha_3^2\}$$

$$= w(r_1)\{\alpha^2 - 2\alpha_1\alpha + \alpha_1^2 + \alpha^2 + 4/3\pi\alpha + 4/9\pi^2 - 2\alpha_1\alpha - 4/3\pi\alpha_1 + \alpha_1^2 + \alpha^2 + 8/3\pi\alpha + 16/9\pi^2 - 2\alpha_1\alpha - 8/3\pi\alpha_1 + \alpha_1^2\}$$

$$+ w(r_2)\{\ldots\} + w(r_3)\{\ldots\}$$

$$= w(r_1)\{3\alpha^2 + 4\pi\alpha - 6\alpha_1\alpha + 20/9\pi^2 - 4\pi\alpha_1 + 3\alpha_1^2\}$$

$$+ w(r_2)\{\ldots\} + w(r_3)\{\ldots\}$$

$$\underline{= w(r_1)\{3\alpha^2 + (4\pi - 6\alpha_1)\alpha + C_1\}}$$

$$\underline{+ w(r_2)\{3\alpha^2 + (4\pi - 6\alpha_2)\alpha + C_2\}}$$

$$\underline{+ w(r_3)\{3\alpha^2 + (4\pi - 6\alpha_3)\alpha + C_3\}}.$$

constants

Thus, we obtain the generalized first derivative

$$\underline{d/d\alpha\, E(\alpha)} = w(r_1)\{6\alpha + 4\pi - 6\alpha_1\} + w(r_2)\{6\alpha + 4\pi - 6\alpha_2\} + w(r_3)\{6\alpha + 4\pi - 6\alpha_3\}$$

$$= 6\alpha(w(r_1) + w(r_2) + w(r_3)) + 4\pi(w(r_1) + w(r_2) + w(r_3)) - 6(w(r_1)\alpha_1 + w(r_2)\alpha_2 + w(r_3)\alpha_3) \underline{\underline{= 0}}$$

! $$\Rightarrow \underline{\alpha = -2/3\pi + (w(r_1)\alpha_1 + w(r_2)\alpha_2 + w(r_3)\alpha_3)} \underline{/(w(r_1) + w(r_2) + w(r_3))}.$$

• **Note.** This formula becomes the formula presented on page **15** (10/25/2023) for $w(r_1) = w(r_2) = w(r_3) = 1$:

$$\underline{\alpha} = -2/3\pi + (1\alpha_1 + 1\alpha_2 + 1\alpha_3)/(1+1+1) \underline{= -2/3 + (\alpha_1 + \alpha_2 + \alpha_3)/3.}$$

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OBJECT AND MATERIAL EIGENFUNCTIONS - Cont'd.

• Laplacian eigenfunctions and neural networks:... For example, one might use a simple weight (function) $w_i = w(r_i) = 1 + (1-r_i)^2$, i.e., a weight that is quadratic in r_i, assigning the value 1 to points on the unit circle and the value 2 to points having the (perpendicular) distance 1 to the unit circle. For this weight one obtains:

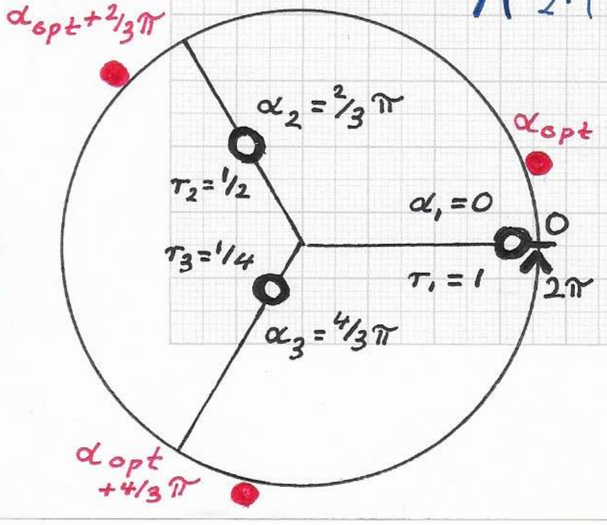
$$\alpha = -\tfrac{2}{3}\pi + \left((1+(1-r_1)^2)\alpha_1 + (1+(1-r_2)^2)\alpha_2 + (1+(1-r_3)^2)\alpha_3\right) / \left(1+(1-r_1)^2 + 1+(1-r_2)^2 + 1+(1-r_3)^2\right)$$

$$= -\tfrac{2}{3}\pi + \left((2-2r_1+r_1^2)\alpha_1 + (2-2r_2+r_2^2)\alpha_2 + (2-2r_3+r_3^2)\alpha_3\right) / \left(6 - 2(r_1+r_2+r_3) + (r_1^2+r_2^2+r_3^2)\right)$$

$$= -\tfrac{2}{3}\pi + \left((2(1-r_1)+r_1^2)\alpha_1 + (2(1-r_2)+r_2^2)\alpha_2 + (2(1-r_3)+r_3^2)\alpha_3\right) / \left(2(3-(r_1+r_2+r_3)) + (r_1^2+r_2^2+r_3^2)\right) \quad (*)$$

• **Note.** Again, this formula must become the formula presented on page 15 (10/25/2023) when the three given points lie on the unit circle, i.e., when $r_1 = r_2 = r_3 = 1$, and it does:

$$\alpha = -\tfrac{2}{3} + \left((0+1)\alpha_1 + (0+1)\alpha_2 + (0+1)\alpha_3\right) / \left(2\cdot(3-3)+3\right) = -\tfrac{2}{3} + (\alpha_1+\alpha_2+\alpha_3)/3.$$



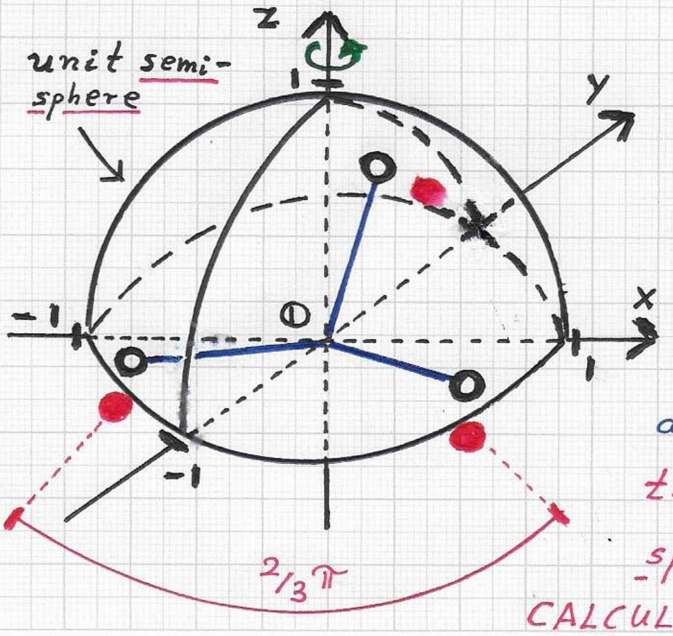
We consider a numerical example for the computation of the optimal value of $\alpha = \alpha_{opt}$, using the above formula (*). The left figure shows the data used. The given α_i-values have optimal, uniform spacing — but the r_i-values are different. ...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... We calculate the α-value for $\alpha_1 = 0$, $\alpha_2 = {}^2/_3\pi$, $\alpha_3 = {}^4/_3\pi$ and $r_1 = 1$, $r_2 = 1/2$, $r_3 = 1/4$:

$$\alpha = -{}^2/_3\pi + \left((2(1-1)+1)\,0 + (2(1-\tfrac{1}{2})+\tfrac{1}{4})\tfrac{2}{3}\pi + (2(1-\tfrac{1}{4})+\tfrac{1}{16})\tfrac{4}{3}\pi\right)$$
$$/\left(2(3-(1+\tfrac{1}{2}+\tfrac{1}{4})) + (1+\tfrac{1}{4}+\tfrac{1}{16})\right)$$
$$= -{}^2/_3\pi + \left(0 + \tfrac{5}{6}\pi + \tfrac{25}{12}\pi\right) / \left(\tfrac{5}{2} + \tfrac{21}{16}\right)$$
$$= -{}^2/_3\pi + \tfrac{35}{12}\pi : \tfrac{61}{16} = -{}^2/_3\pi + \tfrac{35\cdot 16}{12\cdot 61}\pi$$
$$= -{}^2/_3\pi + \tfrac{140}{183}\pi = \tfrac{-122+140}{183}\pi = \tfrac{18}{183}\pi = \tfrac{6}{61}\pi$$

Thus, the resulting optimal triple of uniformly, equidistantly spaced α-values is the triple $\alpha = {}^6/_{61}\pi = .309\ (=17.7^\circ)$, $\alpha + {}^2/_3\pi = {}^{140}/_{183}\pi = 2.403\ (=137.7^\circ)$ and $\alpha + {}^4/_3\pi = {}^{262}/_{183}\pi = 4.498\ (=257.7^\circ)$. The optimal points '●' are shown in the figure on the previous page. — The described methods for an optimal calculation of the α-value needed for optimal placement of the points '●' use a setting where points 'o' and '●' lie in the same plane. The left figure shows a relevant generalization: Here, the points 'o' lie on a unit (semi-) sphere, and the GOAL IS THE CALCULATION OF AN OPTIMAL PLACEMENT OF THE EQUIDISTANTLY SPACED POINTS IN THE XY-PLANE, MINIMIZING A GEODESIC DISTANCE.



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