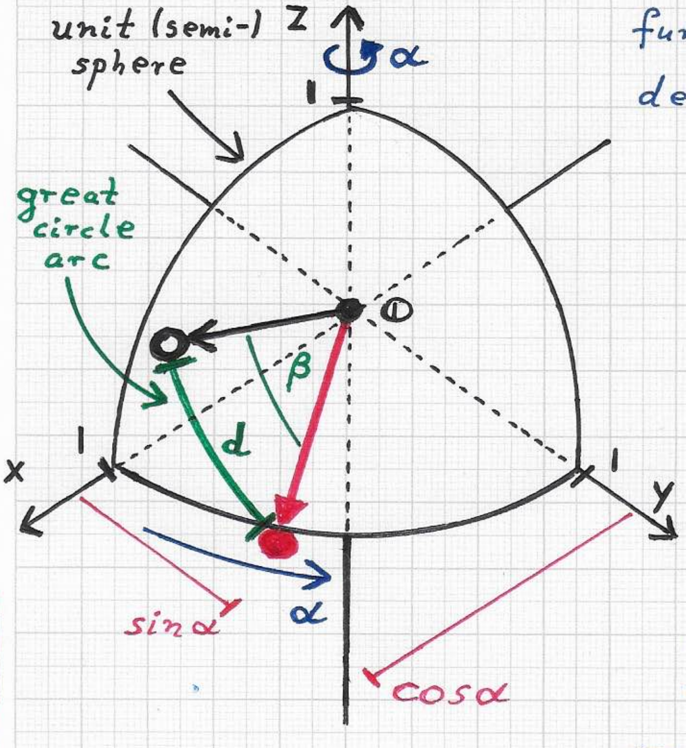
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• **Laplacian eigenfunctions and neural networks:...** A proper definition of point distance in this generalized setting must therefore use GEODESIC DISTANCE OF POINT PAIRS ON THE UNIT SPHERE. As a consequence, we must also generalize the approximation error function that we minimize for optimal placement of the points '●' that lie in a plane and optimally approximate the points 'o' on the unit sphere. Thus, one must establish an overall error



function that involves the definition of geodesic distance. The left figure illustrates the specific setting we are concerned with: We must define the geodesic distance between two points on the unit (semi-) sphere, a given point 'o' with coordinate representation $(x_1, y_1, z_1)^T$ and a point '●' that lies on the unit circle $x^2+y^2=1$ in the xy-plane with coordinate representation $(x, y, 0)^T = (\cos\alpha, \sin\alpha, 0)^T$. The point '●' must eventually be placed in such a way that an overall geodesic-distance-based approximation error function is minimized. The geodesic distance between 'o' and '●' is d.

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OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

- Laplacian eigenfunctions and neural networks:... The geodesic distance of the points 'o' and '●' — both lying on the unit sphere — is the length d of the unique great circle arc that passes through 'o' and '●' and is the shortest path on the unit sphere connecting these points. The value of d is equal to the angle β between 'o' and '●' (viewed as unit positional vectors). For example, the geodesic distance between these two points is MINIMAL when the angle β is MINIMAL — or, alternatively, the value of $\cos\beta$ is MAXIMAL. In order to maximize $\cos\beta$, one must maximize the DOT PRODUCT of the positional vectors 'o' and '●'. The value of this dot product is $\cos\beta = \text{'o'} \cdot \text{'●'} = (x_1, y_1, z_1)^T \cdot (\cos\alpha, \sin\alpha, 0)^T = x_1 \cos\alpha + y_1 \sin\alpha$. If one wanted to minimize the geodesic distance between the points, one would have to determine the proper minimum of the error function

$$E(\alpha) = x_1 \cos\alpha + y_1 \sin\alpha$$

$$\Rightarrow \ d/d\alpha \, E = -x_1 \sin\alpha + y_1 \cos\alpha = 0$$

$$\Rightarrow \quad x_1 \sin\alpha = y_1 \cos\alpha \qquad \{x_1, y_1 \neq 0\}$$

$$\Rightarrow \quad \tan\alpha = y_1 / x_1$$

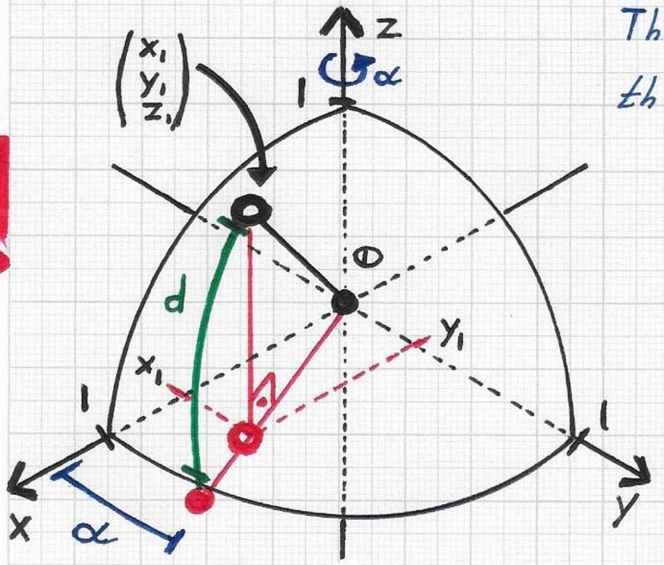
$$\Rightarrow \quad \alpha = \arctan(y_1/x_1) = \alpha_{opt}$$

Thus, one must rotate the vector $\text{'●'} = (1,0,0)^T$ by α_{opt} around the z-axis to obtain its optimal location (minimal distance to 'o'). One must show that α_{opt} is a minimum.

•••

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:… For example, the point $(\sqrt{3}/3, \sqrt{3}/3, \sqrt{3}/3)^T$ generates the value $\alpha = \arctan(\sqrt{3}/3 : \sqrt{3}/3) = \arctan(1) = \pi/4\ (= 45°)$. Rotating $(1,0,0)^T$ by this angle around the z-axis yield the point $(\sqrt{2}/2, \sqrt{2}/2, 0)^T$ — having minimal geodesic distance to $(\sqrt{3}/3, \sqrt{3}/3, \sqrt{3}/3)^T$.



Geometry of algebraic derivation of optimal α-value.

The left figure illustrates the the data and geometry involved in the determination of an optimally placed point '●' (and its defining rotation angle α) on the unit circle in the xy-plane for a given point '○' $= (x_1, y_1, z_1)^T$ on the unit sphere. The perpendicular projection of the point '○' $= (x_1, y_1, z_1)^T$ into the xy-plane yields the point '○' $= (x_1, y_1, 0)^T$ — defining the needed optimal value of the rotation angle via the **tan** function. The value of α is defined by $\tan\alpha = y_1/x_1 \Rightarrow \alpha = \ldots$, where $x_1, y_1 \neq 0$ (special cases). Therefore, the optimal location of the point '●' is obtained by rotating the x-axis basis vector $(1,0,0)^T$ by α around the origin of the xy-plane. This optimal location implies the desired minimal great-circle-arc distance **d** between the points '○' and '●' on the unit sphere. …

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks: … We can now consider the more general problem we are concerned with: Three points 'o' are given on the (semi-) unit sphere, i.e., $\{(x_i, y_i, z_i)^T\}_{i=1}^{3}$ is given with $z_i \geq 0$, for example; and our goal is the optimal placement of three points '•' in the xy-plane and on the unit circle in the xy-plane such that an approximation error function is minimized. The three points '•' are uniformly, equidistantly spaced on the unit circle, with the arc length being $\frac{2}{3}\pi$ between each pair of neighboring points '•'. In other words, the three points have associated angles α, $\alpha + \frac{2}{3}\pi$ and $\alpha + \frac{4}{3}\pi$, where the initial value of α is understood relative to a designated location $\alpha = 0$ on the unit circle. Thus, one must optimize the value of ONLY ONE VARIABLE: α. THE VALUE OF α IS OPTIMAL WHEN THE ERROR FUNCTION HAS MINIMAL VALUE — WHERE THE ERROR FUNCTION CONSIDERS A SUM OF THE THREE GEODESIC DISTANCES BETWEEN THE THREE POINTS 'o' AND THEIR THREE POINTS '•' (IDEALLY ONLY INVOLVING THREE GEODESIC DISTANCES FOR EXACTLY THREE PAIRS OF ASSOCIATED PAIRS OF A 'o' AND A '•' POINT). The result of this optimization is a triple of '•' points that represents a best approximation of the the given triple of 'o' points. …

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... We can now define this more general optimization problem algebraically.

• **Note.** Solving the optimization problem involves the use of formulas for the **sin** and **cos** functions for sums of angles. The formulas — based on Euler's formula — are:

$$\sin(\alpha_1 +/- \alpha_2) = \sin\alpha_1 \cos\alpha_2 +/- \cos\alpha_1 \sin\alpha_2 ;$$
$$\cos(\alpha_1 +/- \alpha_2) = \cos\alpha_1 \cos\alpha_2 -/+ \sin\alpha_1 \sin\alpha_2 .$$

Further, since the points '●' have an "angle spacing" of $2/3\,\pi$ between each neighboring pair, we must use the following values for the univariate optimization:

$$\sin(2/3\,\pi) = \sqrt{3}/2,\ \sin(4/3\,\pi) = -\sqrt{3}/2;\ \cos(2/3\,\pi) = -1/2,\ \cos(4/3\,\pi) = -1/2.$$

We can now define the error function to be minimized:

$$E(\alpha) = x_1\cos\alpha + y_1\sin\alpha + x_2\cos(\alpha + 2/3\,\pi) + y_2\sin(\alpha + 2/3\,\pi) + $$
$$+ x_3\cos(\alpha + 4/3\,\pi) + y_3\sin(\alpha + 4/3\,\pi)$$

$$= x_1\cos\alpha + y_1\sin\alpha +$$
$$+ x_2(\cos\alpha\cos 2/3\,\pi - \sin\alpha\sin 2/3\,\pi) + y_2(\sin\alpha\cos 2/3\,\pi + \cos\alpha\sin 2/3\,\pi) +$$
$$+ x_3(\cos\alpha\cos 4/3\,\pi - \sin\alpha\sin 4/3\,\pi) + y_3(\sin\alpha\cos 4/3\,\pi + \cos\alpha\sin 4/3\,\pi)$$

$$= x_1\cos\alpha + y_1\sin\alpha +$$
$$+ x_2(\cos\alpha\cdot(-1/2) - \sin\alpha\cdot(\sqrt{3}/2)) + y_2(\sin\alpha\cdot(-1/2) + \cos\alpha\cdot(\sqrt{3}/2)) +$$
$$+ x_3(\cos\alpha\cdot(-1/2) - \sin\alpha\cdot(-\sqrt{3}/2)) + y_3(\sin\alpha\cdot(-1/2) + \cos\alpha\cdot(-\sqrt{3}/2))$$

$$= \sin\alpha\cdot(y_1 - \sqrt{3}/2\,x_2 - 1/2\,y_2 + \sqrt{3}/2\,x_3 - 1/2\,y_3) +$$
$$+ \cos\alpha\cdot(x_1 - 1/2\,x_2 + \sqrt{3}/2\,y_2 - 1/2\,x_3 - \sqrt{3}/2\,y_3)$$

$$= \sin\alpha\cdot(y_1 + \sqrt{3}/2\,(x_3 - x_2) - 1/2\,(y_3 + y_2)) + \cos\alpha\cdot(x_1 + \sqrt{3}/2\,(y_2 - y_3) - 1/2\,(x_2 + x_3))$$

$$= \mathbf{A}\sin\alpha + \mathbf{B}\cos\alpha$$

$$\Rightarrow \frac{d}{d\alpha}E(\alpha) = \mathbf{A}\cos\alpha - \mathbf{B}\sin\alpha = 0 \Rightarrow \frac{\sin\alpha}{\cos\alpha} = \tan\alpha = \frac{\mathbf{A}}{\mathbf{B}}$$

$$\Rightarrow \alpha = \arctan(\mathbf{A}/\mathbf{B}) = \alpha_{opt}$$

...