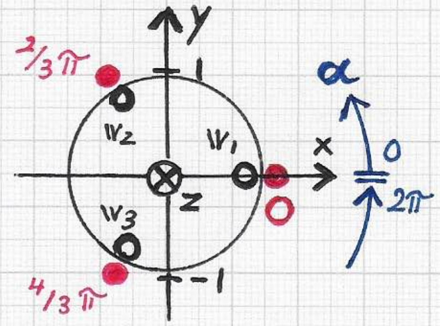
Bernd Hamann 11/2/2023

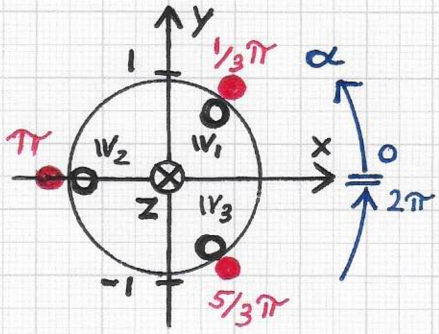
Stratovan

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks: ...



Given points 'o' are in optimal position; the calculated parameter values (α-values) reproduce these points exactly – as points '•'.



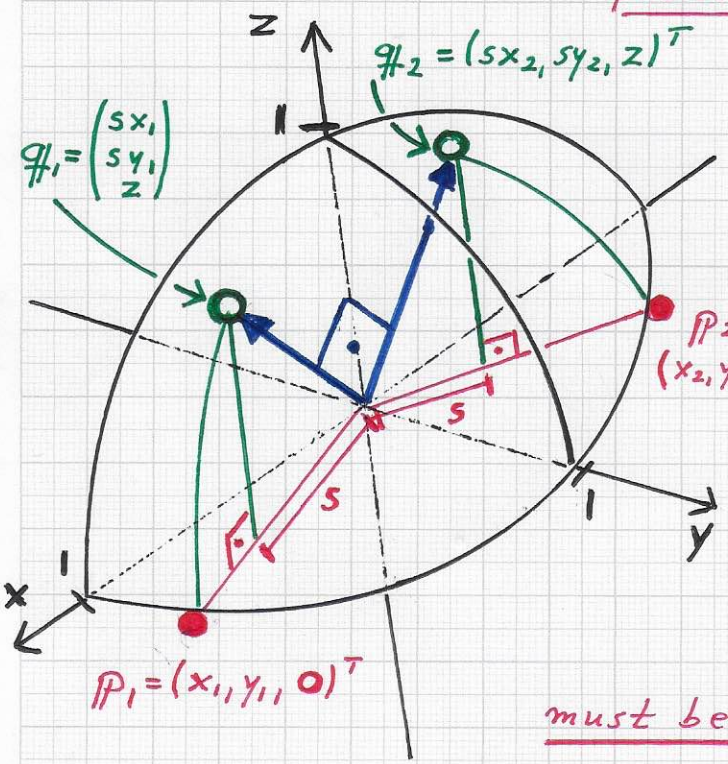
Second scenario where given points 'o' are reproduced exactly by calculated points '•'.

We consider two simple examples using this method for optimal α-value calculation. The two left figures sketch two cases – "Looking down the z-axis into the xy-plane." In the first case, the three given points/vectors are already in "perfect positions": $\mathbb{W}_1 = (1, 0, 0)^T$, $\mathbb{W}_2 = (-\sqrt{3}/2, \frac{1}{2}, 0)^T$ and $\mathbb{W}_3 = (-\sqrt{3}/2, -\frac{1}{2}, 0)^T$. Thus, we obtain $A = 0 + \frac{\sqrt{3}}{2} \cdot 0 - \frac{1}{2} \cdot 0 = 0$ and $B = 1 + \frac{\sqrt{3}}{2} \cdot 1 - \frac{1}{2} \cdot (-\sqrt{3}) = 1 + \sqrt{3}$. Thus, $\tan\alpha = 0/(1+\sqrt{3}) = 0 \Rightarrow \alpha = 0$. The resulting optimal triple of **α-values** for the three points/vectors '•' is $(0, \frac{2}{3}\pi, \frac{4}{3}\pi)$. In the second case, the three given points/vectors are $\mathbb{W}_1 = (\frac{1}{2}, \frac{\sqrt{3}}{2}, 0)^T$, $\mathbb{W}_2 = (-1, 0, 0)^T$ and $\mathbb{W}_3 = (\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0)^T$.

Thus, we obtain $A = \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{3}{2} - \frac{1}{2} \cdot (-\frac{\sqrt{3}}{2}) = \frac{\sqrt{3}}{2} + \frac{3}{4}\sqrt{3} + \frac{1}{4}\sqrt{3} = \frac{3}{2}\sqrt{3}$ and $B = \frac{1}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{1}{2} \cdot (-\frac{1}{2}) = \frac{1}{2} + \frac{3}{4} + \frac{1}{4} = \frac{3}{2}$. Therefore, $\tan\alpha = \frac{3}{2}\sqrt{3} / \frac{3}{2} = \sqrt{3} \Rightarrow \alpha = \arctan(\sqrt{3}) = \frac{1}{3}\pi$. In this case, the resulting optimal triple of the three optimally placed points/vectors '•' is $(\frac{1}{3}\pi, \pi, \frac{5}{3}\pi)^T$. Of course, the method handles points/vectors 'o' in any position on the (semi-)sphere. ...

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• **Laplacian eigenfunctions and neural networks: ...**



Once the optimally placed points '•' are known in the xy-plane ($z=0$), one must calculate the points/positional vectors 'o' shown in the left figure. The points/positional vectors p_1 and p_2 are NOT orthogonal to each other. Since ORTHONORMALITY must be enforced for this "pair," one must "move the unit positional vectors p_1 and p_2 upwards, by moving them to a 'height z' on the shown unit (semi-)sphere until they mapped to the points/positional vectors q_1 and q_2 that ARE ORTHOGONAL."

We call the needed scaling factor s and needed height above the xy-plane z. We can compute q_1 and q_2 as follows:

$$q_1 = (sx_1, sy_1, z)^T, \quad q_2 = (sx_2, sy_2, z)^T$$

$\Rightarrow$ orthogonality: $q_1 \cdot q_2 = s^2(x_1 x_2 + y_1 y_2) + z^2 = 0$

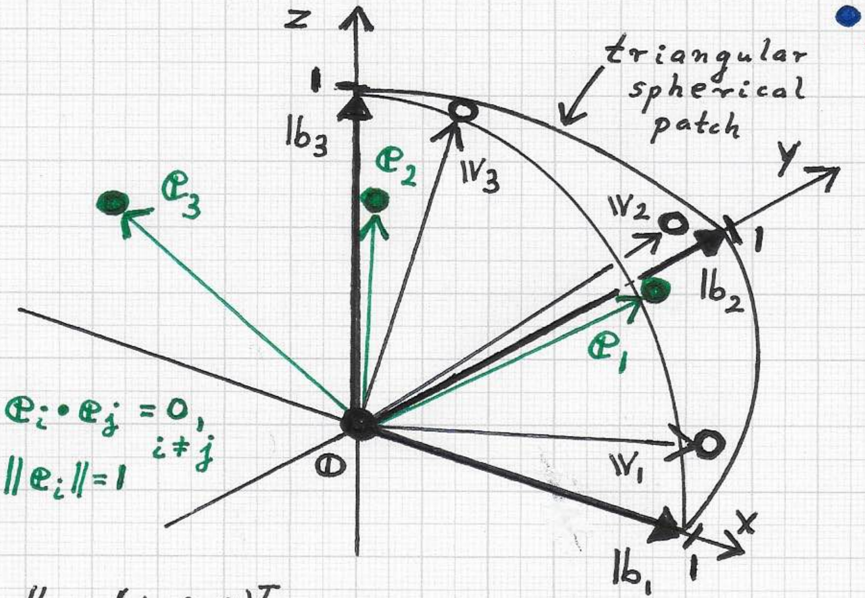
$\Rightarrow$ using $s^2 + z^2 = 1$: $(1-z^2)\, p_1 \cdot p_2 + z^2 = 0 \Rightarrow z^2 = (p_1 \cdot p_2)/(p_1 \cdot p_2 - 1)$

$$\Rightarrow \quad z = +/- \left((p_1 \cdot p_2)/(p_1 \cdot p_2 - 1) \right)^{1/2}, \quad s = (1 - z^2)^{1/2}.$$

The resulting pair(s) of orthonormal points/vectors q_1, q_2 is the needed optimal orthonormal vector pair. ...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks: … • **Note.** The values of z and s derived on the previous page must be applied to all points $\mathbb{p}_i = (x_i, y_i, 0)^T$, $i=1\dots3$, lying on the unit circle in the xy-plane ($z=0$), with uniform, equidistant circular-arc spacing between neighbors. The result is the desired "optimal" orthonormal basis, defined by the three vectors $\mathbb{q}_i = (sx_i, sy_i, z)^T$, $i=1\dots3$.



$\mathbb{e}_i \cdot \mathbb{e}_j = 0$, $i \neq j$

$\|\mathbb{e}_i\| = 1$

• **Example.** We now apply this method to the data/point/vector configuration described on pages 4–5 (10/19–20/2023). The left figure shows the relevant data, together with the coordinate values of all nine points/positional vectors. For this simple example, the optimization approach must map the given vector $\mathbb{w}_i$ to $\mathbb{b}_i$, $i=1\dots3$.

$\mathbb{b}_1 = (1,0,0)^T$

$\mathbb{b}_2 = (0,1,0)^T$

$\mathbb{b}_3 = (0,0,1)^T$

$\mathbb{w}_1 = \sqrt{2}\,(2/3, 1/6, 1/6)^T$

$\mathbb{w}_2 = \sqrt{2}\,(1/6, 2/3, 1/6)^T$

$\mathbb{w}_3 = \sqrt{2}\,(1/6, 1/6, 2/3)^T$

$\mathbb{e}_1 = \sqrt{3}\,(1/3, 1/3, 1/3)^T$

$\mathbb{e}_2 = \sqrt{2}\,(-1/2, 1/2, 0)^T$

$\mathbb{e}_3 = \sqrt{6}\,(-1/6, -1/6, 1/3)^T$

"Standard" global orthonormal basis $\{\mathbb{b}_i\}$; given set of unit, non-orthogonal vectors $\mathbb{w}_i$; derived orthonormal eigenbasis $\{\mathbb{e}_i\}$; $i=1\dots3$.

•••

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

- **Laplacian eigenfunctions and neural networks:...** Since the line/axis defined by vector $\mathbb{e}_1$ — associated with the largest eigenvalue implied by the given vectors $\mathbb{v}_i$ and resulting PCA — specifies the needed "rotation axis," the first step must project the given points/vectors $\mathbb{v}_i$ into the proper plane. Thus, $\mathbb{e}_1$ defines the unit normal vector of the plane that passes through the origin and that $\mathbb{v}_i$ must be projected into (perpendicularly). Further, the orthonormal vectors $\mathbb{e}_2$ and $\mathbb{e}_3$ define a basis for this plane (with the global origin $\mathbb{O}$ also being the origin of the plane's orthonormal coordinate system). Once $\mathbb{v}_1, \mathbb{v}_2$ and $\mathbb{v}_3$ have been projected into the plane and are represented relative to the coordinate system $\{\mathbb{O}, \mathbb{e}_2, \mathbb{e}_3\}$, they must be normalized — such that the resulting points lie on the unit circle in this 2D coordinate system in the projection plane. We discuss this example step-by-step.

Step I. Projecting (orthographically) $\mathbb{v}_1, \mathbb{v}_2$ and $\mathbb{v}_3$ into the e_2e_3-plane. The normal vector of the e_2e_3-plane (passing through the origin) is $\mathbb{e}_1$, i.e., $\mathbb{e}_1 = \sqrt{3}\,(1/3, 1/3, 1/3)^T$. Thus, the projection plane is defined implicitly as $\sqrt{3}/3\,(x+y+z)=0$. The parametric definition of the line passing through $\mathbb{v}_i$ and having $\mathbb{e}_1$ as its direction vector is $\mathbb{x}_i(t) = \mathbb{v}_i + t\,\mathbb{e}_1$, $i=1,2,3$. ...

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■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks: ... One obtains the needed value of the parameter t ("Line-plane intersection") by inserting the $x_i(t)$, $y_i(t)$ and $z_i(t)$ expressions of $\mathbf{x}_i(t)$ into the implicit plane equation and solving for t:

$$\mathbf{x}_i(t) = (x_i(t), y_i(t), z_i(t))^T = (x_i, y_i, z_i)^T + t\sqrt{3}\,(1/3, 1/3, 1/3)^T$$

$$\Rightarrow \sqrt{3}/3\,(x_i(t)+y_i(t)+z_i(t)) = \sqrt{3}/3\,(\sqrt{2} + t\sqrt{3}) = \sqrt{6}/3 + t$$

(where $\mathbf{v}_i = (x_i, y_i, z_i)^T$ and $x_i + y_i + z_i = \sqrt{2}$, $i = 1,2,3$)

$$\Rightarrow \sqrt{6}/3 + t = 0 \Rightarrow t = -\sqrt{6}/3 .$$

Using this t-value yield the three projection points:

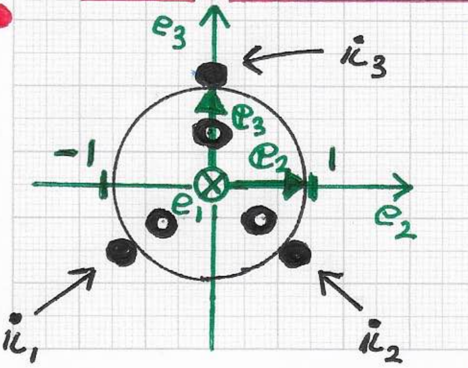
$$\mathbf{r}_1 = \mathbf{v}_1 - \sqrt{6}/3\,\sqrt{3}\,(1/3, 1/3, 1/3)^T = \sqrt{2}\,(2/3, 1/6, 1/6)^T - \sqrt{2}\,(1/3, 1/3, 1/3)^T$$
$$= \sqrt{2}\,(2/3 - 1/3, 1/6 - 1/3, 1/6 - 1/3)^T = \sqrt{2}\,(1/3, -1/6, -1/6)^T$$

$$\mathbf{r}_2 = \mathbf{v}_2 - \sqrt{6}/3\,\sqrt{3}\,(1/3, 1/3, 1/3)^T = \sqrt{2}\,(1/6, 2/3, 1/6)^T - \sqrt{2}\,(1/3, 1/3, 1/3)^T$$
$$= \sqrt{2}\,(1/6 - 1/3, 2/3 - 1/3, 1/6 - 1/3)^T = \sqrt{2}\,(-1/6, 1/3, -1/6)^T$$

$$\mathbf{r}_3 = \mathbf{v}_3 - \sqrt{6}/3\,\sqrt{3}\,(1/3, 1/3, 1/3)^T = \sqrt{2}\,(1/6, 1/6, 2/3)^T - \sqrt{2}\,(1/3, 1/3, 1/3)^T$$
$$= \sqrt{2}\,(1/6 - 1/3, 1/6 - 1/3, 2/3 - 1/3)^T = \sqrt{2}\,(-1/6, -1/6, 1/3)^T$$

(where $\sqrt{2}/3 = .47$ and $\sqrt{2}/6 = .24$)

Step II. Expressing the three projection points in the $e_2 e_3$-plane via two coordinates, relative to $\mathbf{e}_2$ and $\mathbf{e}_3$.



The second step first represents the points $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ in the $e_2 e_3$-plane via two coordinates and then maps these intermediate points/positional vectors, using normalization, to points on the unit circle $e_2^2 + e_3^2 = 1$.

The above figure shows the points '○' and '●' that result, called $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$. ...