■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• **Laplacian eigenfunctions and neural networks:...** The figure on the previous page provides a "top-down view," with the viewer on the $\mathbf{e}_1$-axis, looking into the $\mathbf{e}_2\mathbf{e}_3$-plane. We calculate the coordinates of the points '○' (not normalized, not on unit circle) and '●' (normalized, on unit circle), called $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$ for simplicity of notation, in the $\mathbf{e}_2\mathbf{e}_3$-plane. Thus, the 2D coordinates of the points '○' are obtained via computing dot products:

$$\mathbf{u}_1 = (\mathbf{u}_1 \cdot \mathbf{e}_2)\,\mathbf{e}_2 + (\mathbf{u}_1 \cdot \mathbf{e}_3)\,\mathbf{e}_3 \;,$$
$$\mathbf{u}_2 = (\mathbf{u}_2 \cdot \mathbf{e}_2)\,\mathbf{e}_2 + (\mathbf{u}_2 \cdot \mathbf{e}_3)\,\mathbf{e}_3 \;,$$
$$\mathbf{u}_3 = (\mathbf{u}_3 \cdot \mathbf{e}_2)\,\mathbf{e}_2 + (\mathbf{u}_3 \cdot \mathbf{e}_3)\,\mathbf{e}_3 \;.$$

The values of the six dot products are:

$$\mathbf{u}_1 \cdot \mathbf{e}_2 = \sqrt{2}\,(1/3, -1/6, -1/6)^T \cdot \sqrt{2}\,(-1/2, 1/2, 0)^T = -1/2 \;,$$
$$\mathbf{u}_1 \cdot \mathbf{e}_3 = \sqrt{2}\,(1/3, -1/6, -1/6)^T \cdot \sqrt{6}\,(-1/6, -1/6, 1/3)^T = -\sqrt{3}/6 \;,$$
$$\mathbf{u}_2 \cdot \mathbf{e}_2 = \sqrt{2}\,(-1/6, 1/3, -1/6)^T \cdot \sqrt{2}\,(-1/2, 1/2, 0)^T = 1/2 \;,$$
$$\mathbf{u}_2 \cdot \mathbf{e}_3 = \sqrt{2}\,(-1/6, 1/3, -1/6)^T \cdot \sqrt{6}\,(-1/6, -1/6, 1/3)^T = -\sqrt{3}/6 \;,$$
$$\mathbf{u}_3 \cdot \mathbf{e}_2 = \sqrt{2}\,(-1/6, -1/6, 1/3)^T \cdot \sqrt{2}\,(-1/2, 1/2, 0)^T = 0 \;,$$
$$\mathbf{u}_3 \cdot \mathbf{e}_3 = \sqrt{2}\,(-1/6, -1/6, 1/3)^T \cdot \sqrt{6}\,(-1/6, -1/6, 1/3)^T = \sqrt{3}/3 \;.$$

The 2D tuples $(\mathbf{u}_j \cdot \mathbf{e}_2, \mathbf{u}_j \cdot \mathbf{e}_3)^T$, $j = 1, 2, 3$, define the desired 2D representation/coordinates of $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$. The three points – their positional vectors – have the associated vector magnitude $((-1/2)^2 + (-\sqrt{3}/6)^2)^{1/2} = ((1/2)^2 + (-\sqrt{3}/6)^2)^{1/2} = (0^2 + (\sqrt{3}/3)^2)^{1/2} = \sqrt{3}/3$. By dividing the coordinate values of $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$ by this magnitude, we obtain the points on the unit circle ($\mathbf{e}_2\mathbf{e}_3$-plane).

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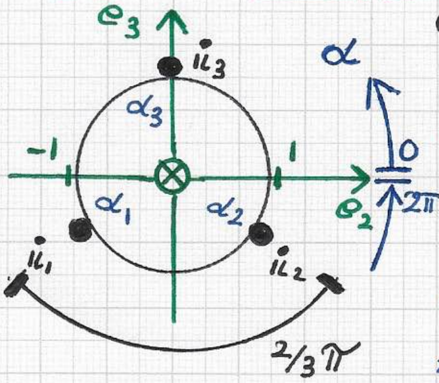
OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... This normalization step yields the three points '•' on the unit cirle (division by $\sqrt{3}/3 \hat{=}$ multiplication by $\sqrt{3}$):

$$\mathbf{u}_1 = \sqrt{3}\ (-1/2, -\sqrt{3}/6)^T = (-\sqrt{3}/2, -1/2)^T,$$
$$\mathbf{u}_2 = \sqrt{3}\ (1/2, -\sqrt{3}/6)^T = (\sqrt{3}/2, -1/2)^T,$$
$$\mathbf{u}_3 = \sqrt{3}\ (0, \sqrt{3}/3)^T = (0, 1)^T.$$

Step III. Placing points optimally on unit circle in the e_2e_3-plane. In this case, the three points '•' are already placed optimally on the circle $e_2^2 + e_3^2 = 1$, as they are equidistantly spaced, having uniform distance $2/3\pi$ from each other, see left figure. Therefore, the formal algebraic approach must reproduce this point placement exactly. The angles associated with $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$ are α_1, α_2 and α_3, respectively – as shown in the figure. Their values are $\alpha_1 = 7/6\,\pi$, $\alpha_2 = 11/6\,\pi$ and $\alpha_3 = 3/6\,\pi$. Using the formula on page 15 (10/25/2023) for the optimal angle α_{opt}, one obtains:

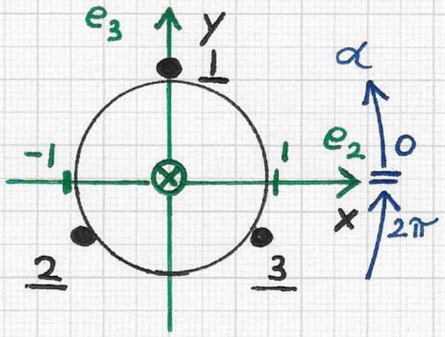


$$\alpha_{opt} = -2/3\pi + (\alpha_1 + \alpha_2 + \alpha_3)/3 = -2/3\pi + (7/6\pi + 11/6\pi + 3/6\pi)/3$$
$$= -2/3\pi + (21/6\,\pi)/3 = -2/3\pi + (7/2\pi)/3 = -4/6\pi + 7/6\pi = \pi/2.$$

Thus, the optimal angle triple is $\pi/2, 7/6\pi, 11/6\pi$ – as expected. This angle triple is the same angle triple associated with $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$, implying that these three points are reproduced by this optimization step. ...

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks: ...



	x	y
1	0	1
2	$-\sqrt{3}/2$	$-1/2$
3	$\sqrt{3}/2$	$-1/2$

{ • **Note.** One can also use the more general approach that is defined by the formula provided on page **25** (10/31/2023) for the computation of the α-value. In this case, it is appropriate to "order" the three points '•', relative to the counter-clockwise oriented α-parameter that starts at 0 and ends at 2π – beginning at point $(1,0)^T$. The top-left figure provides an illustration and shows the three points with indices 1, 2 and 3; the table lists the coordinate values – keeping in mind that e_2 corresponds to x and e_3 corresponds to y. For these values one obtains:

$$A = 1 + \sqrt{3}/2\,(\sqrt{3}/2 + \sqrt{3}/2) - \tfrac{1}{2}(-\tfrac{1}{2} - \tfrac{1}{2}) = 1 + \tfrac{3}{2} + \tfrac{1}{2} = 3,$$

$$B = 0 + \sqrt{3}/2\,(-1/2 + 1/2) - \tfrac{1}{2}(-\sqrt{3}/2 + \sqrt{3}/2) = 0 + 0 - 0 = 0;$$

$$\tan\alpha = A/B = 3/0 = +\infty,$$

$$\alpha = \arctan(+\infty) = \pi/2 \quad (= 90°).$$

Thus, the resulting optimal triple of angle values is – as expected and in agreement with the other, specialized optimization method – $\pi/2,\ 7/6\,\pi,\ 11/6\,\pi$.

This numerical example concerns a special case: $A/B = 3/0$, and one must properly interpret $\arctan(+\infty)$. }

...

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... **Step IV.** Enforcing orthonormality and defining the optimal 3D basis vectors. The last step concerns the mapping of the points $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$ (provided at the top of page 7 from 11/6/2023), which are placed optimally in the $e_2 e_3$-plane, to points $\#_1$, $\#_2$ and $\#_3$ in 3D $e_1 e_2 e_3$-space. The points $\#_1$, $\#_2$ and $\#_3$ are constructed in a manner that ensures orthonormality of their associated positional vectors in 3D space, see page 2 from 11/2/2023. **As** described, one must compute the values of inner/dot products and a scaling factor. First, the needed dot product values are:

$$\mathbf{u}_1 \bullet \mathbf{u}_2 = (-\sqrt{3}/2, -1/2)^T \bullet (\sqrt{3}/2, -1/2)^T = -1/2 ,$$
$$\mathbf{u}_1 \bullet \mathbf{u}_3 = (-\sqrt{3}/2, -1/2)^T \bullet (0, 1)^T = -1/2 ,$$
$$\mathbf{u}_2 \bullet \mathbf{u}_3 = (\sqrt{3}/2, -1/2)^T \bullet (0, 1)^T = -1/2 .$$

Since these three values are equal, one obtains one unique **z-value**; it is

$$z = +((\mathbf{u}_1 \bullet \mathbf{u}_2)/(\mathbf{u}_1 \bullet \mathbf{u}_2 - 1))^{1/2}$$
$$= +((\mathbf{u}_1 \bullet \mathbf{u}_3)/(\mathbf{u}_1 \bullet \mathbf{u}_3 - 1))^{1/2}$$
$$= +((\mathbf{u}_2 \bullet \mathbf{u}_3)/(\mathbf{u}_2 \bullet \mathbf{u}_3 - 1))^{1/2}$$
$$= +((-1/2)/(-3/2))^{1/2} = \sqrt{3}/3 .$$

Second, one computes the unique value of the scaling factor:

$$s = (1 - z^2)^{1/2} = (1 - 1/3)^{1/2} = (2/3)^{1/2} = \sqrt{6}/3 .$$

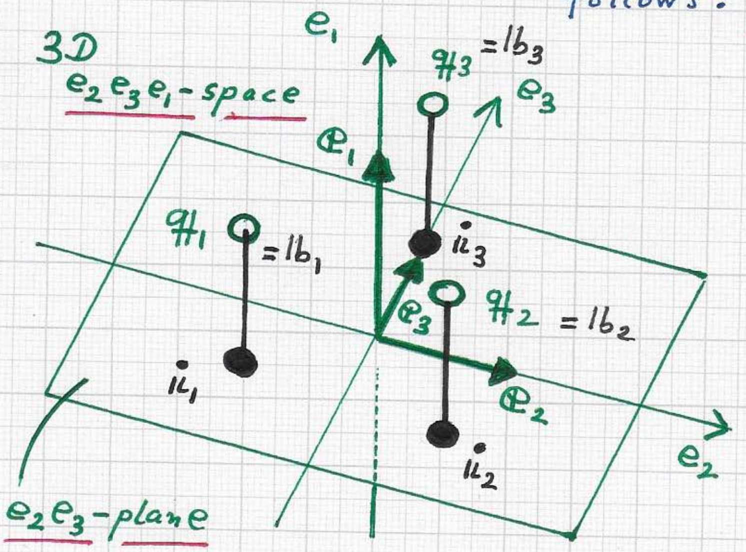
We can now calculate the coordinate values of the three desired orthonormal basis vectors. •••

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

- **Laplacian eigenfunctions and neural networks:...**

We compute $\mathbf{q}_1$, $\mathbf{q}_2$ and $\mathbf{q}_3$ as follows:

$$\mathbf{q}_1 = \left(s\left(-\tfrac{\sqrt{3}}{2}\right), s\left(-\tfrac{1}{2}\right), z\right)^T = \left(\tfrac{\sqrt{6}}{3}\left(-\tfrac{\sqrt{3}}{2}\right), \tfrac{\sqrt{6}}{3}\left(-\tfrac{1}{2}\right), \tfrac{\sqrt{3}}{3}\right)^T = \left(-\tfrac{\sqrt{2}}{2}, -\tfrac{\sqrt{6}}{6}, \tfrac{\sqrt{3}}{3}\right)^T.$$



The coordinates of this point/positional vector are relative to the system with the orthonormal basis vectors $\mathbf{e}_2$, $\mathbf{e}_3$ and $\mathbf{e}_1$ (in this order), see left figure.

Therefore, the representation of $\mathbf{q}_1$ relative to the global ("world") coordinate system is given as

$$\mathbf{q}_1 = -\tfrac{\sqrt{2}}{2}\mathbf{e}_2 - \tfrac{\sqrt{6}}{6}\mathbf{e}_3 + \tfrac{\sqrt{3}}{3}\mathbf{e}_1 = -\tfrac{\sqrt{2}}{2}\left(-\tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}}{2}, 0\right)^T - \tfrac{\sqrt{6}}{6}\left(-\tfrac{\sqrt{6}}{6}, -\tfrac{\sqrt{6}}{6}, \tfrac{\sqrt{6}}{3}\right)^T + \tfrac{\sqrt{3}}{3}\left(\tfrac{\sqrt{3}}{3}, \tfrac{\sqrt{3}}{3}, \tfrac{\sqrt{3}}{3}\right)^T$$

$$= \left(\tfrac{1}{2}+\tfrac{1}{6}+\tfrac{1}{3}, -\tfrac{1}{2}+\tfrac{1}{6}+\tfrac{1}{3}, 0-\tfrac{1}{3}+\tfrac{1}{3}\right)^T = (1,0,0)^T = \mathbf{b}_1.$$

Thus, the method has correctly reproduced the vector $\mathbf{b}_1$.

The other resulting orthonormal basis vectors are:

$$\mathbf{q}_2 = \left(\tfrac{\sqrt{6}}{3}\left(\tfrac{\sqrt{3}}{2}\right), \tfrac{\sqrt{6}}{3}\left(-\tfrac{1}{2}\right), \tfrac{\sqrt{3}}{3}\right)^T = \left(\tfrac{\sqrt{2}}{2}, -\tfrac{\sqrt{6}}{6}, \tfrac{\sqrt{3}}{3}\right)^T$$

$$= \tfrac{\sqrt{2}}{2}\mathbf{e}_2 - \tfrac{\sqrt{6}}{6}\mathbf{e}_3 + \tfrac{\sqrt{3}}{3}\mathbf{e}_1 = \left(-\tfrac{1}{2}+\tfrac{1}{6}+\tfrac{1}{3}, \tfrac{1}{2}+\tfrac{1}{6}+\tfrac{1}{3}, 0-\tfrac{1}{3}+\tfrac{1}{3}\right)^T$$

$$= (0, 1, 0)^T = \mathbf{b}_2,$$

$$\mathbf{q}_3 = \left(\tfrac{\sqrt{6}}{3}\cdot 0, \tfrac{\sqrt{6}}{3}\cdot 1, \tfrac{\sqrt{3}}{3}\right)^T = \left(0, \tfrac{\sqrt{6}}{3}, \tfrac{\sqrt{3}}{3}\right)^T$$

$$= 0\,\mathbf{e}_2 + \tfrac{\sqrt{6}}{3}\mathbf{e}_3 + \tfrac{\sqrt{3}}{3}\mathbf{e}_1 = \left(0-\tfrac{1}{3}+\tfrac{1}{3}, 0-\tfrac{1}{3}+\tfrac{1}{3}, 0+\tfrac{2}{3}+\tfrac{1}{3}\right)^T$$

$$= (0, 0, 1)^T = \mathbf{b}_3.$$

This detailed numerical example demonstrates that **steps I–IV** generate the expected result exactly. ...