OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• **Laplacian eigenfunctions and neural networks:...** In summary, one can describe the process of "optimal orthonormalization" as a sequence of the following algorithmic steps:

0) For the non-orthogonal 3D basis $\{\mathbb{v}_1, \mathbb{v}_2, \mathbb{v}_3\}$ construct the implied orthonormal eigenbasis $\{\mathbb{e}_1, \mathbb{e}_2, \mathbb{e}_3\}$, by using principal components analysis (PCA) and calling the normalized eigenvector associated with the largest eigenvalue $\mathbb{e}_1$.

1) Perform an orthographic projection of the unit vectors $\mathbb{v}_1$, $\mathbb{v}_2$ and $\mathbb{v}_3$ into the e_2e_3-plane that has $\mathbb{e}_1$ as its unit normal vector, thus using $\mathbb{e}_1$ as direction vector for the projection.

2) Express/represent the generated points/positional vectors in the 2D e_2e_3-plane relative to the orthonormal basis vectors $\mathbb{e}_2$ and $\mathbb{e}_3$ and "normalize" the points/positional vectors by placing them onto the unit circle $e_2^2 + e_3^2 = 1$.

3) Calling the points/positional vectors resulting from the last step $\mathbb{u}_1$, $\mathbb{u}_2$ and $\mathbb{u}_3$, "re-position" them on the circle $e_2^2 + e_3^2 = 1$ optimally, by moving them by minimal amounts (angles, arc lengths) to obtain a uniform, equidistant spacing for them on the circle $e_2^2 + e_3^2 = 1$.

...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... 4) Constructing an optimal 3D orthonormal basis $\{q_1, q_2, q_3\}$, by mapping the optimally positioned points/positional vectors $\bar{u}_1$, $\bar{u}_2$ and $\bar{u}_3$ onto the unit sphere in 3D "world space."

These steps succinctly define the overall algorithm for the establishment of an optimal 3D orthonormal basis that approximates – in a defined best-approximation sense – a given 3D non-orthogonal basis. One can explore how to potentially generalize this 3D construction to the 4D, 5D and n-dimensional setting. Rotation of points/positional vectors is a crucial tranformation in the described 3D case, and it is thus important to understand the generalization of rotation to the 4D, 5D and n-dimensional case. Informally, one can describe a general rotation as follows:

> In n-dimensional space ($n \geq 2$), a rotation affects the coordinates of points/positional vectors in a 2-dimensional sub-space while it does not change the coordinates of points/positional vectors in the dual, complementary $(n-2)$-dimensional sub-space.

This description properly captures rotations in the plane and 3D space. ...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• **Laplacian eigenfunctions and neural networks:...** In the following, we review some of the related geometrical, algebraic and combinatorial aspects when defining and performing rotations in n-dimensional space. For example, one must understand the relationships between matrix algebra, determinants, basis vectors and orientation of coordinate systems (e.g., "right-handed" and "left-handed" coordinate systems in 3D space). First, we review rotations in the plane and in 3D space before discussing rotations in 4D space.

• **Note.** A rotation, with associated matrix $\mathbf{R}$, is an orthogonal transformating, mapping orthogonal vectors to orthogonal vectors; its associated matrix satisfies $\mathbf{R}^{-1} = \mathbf{R}^T$; a rotation is PROPER or PURE when $\det(\mathbf{R}) = 1$. (A so-called IMPROPER rotation is the result of concatenating a proper rotation and a reflection. The determinant of the matrix associated with an improper rotation has the value -1.)

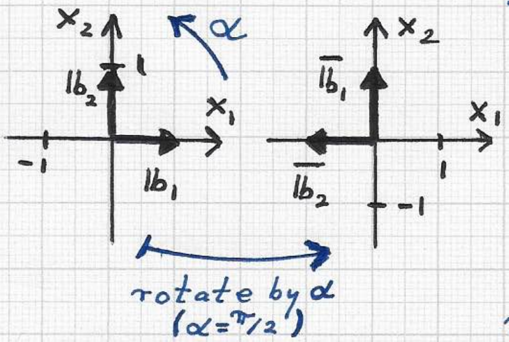
Correct and consistant orientation and indexing (of basis vectors and coordinate system axes) and the appropriate position of the "$-\sin(\alpha)$" component in the rotation matrix are crucially important. These aspects are briefly reviewed in the following.

...

Stratovan

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

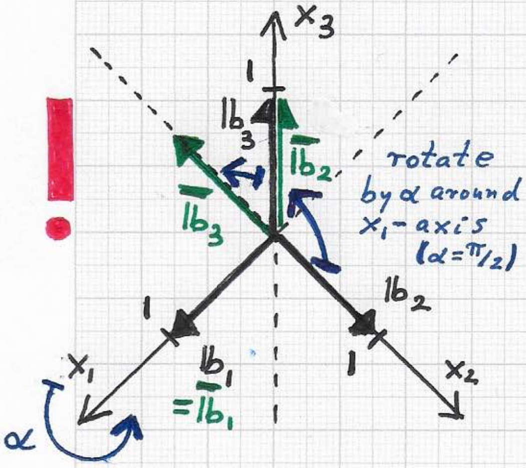
• Laplacian eigenfunctions and neural networks:...



First, we consider the fundamental rotation in the plane, in 2D space. The left figure captures the essential aspects. The two orthonormal basis vectors $\mathbb{b}_1$ and $\mathbb{b}_2$ — with associated axes x_1 and x_2, respectively — are rotated around the origin. Rotating $\mathbb{b}_1$ and $\mathbb{b}_2$ by $\pi/2$ (90°) yields the orthonormal vectors $\overline{\mathbb{b}}_1$ and $\overline{\mathbb{b}}_2$, respectively. One obtains $\overline{\mathbb{b}}_1 = \mathbb{b}_2$ and $\overline{\mathbb{b}}_2 = -\mathbb{b}_1$. The associated rotation matrix R has "$-\sin(\alpha)$" as its upper-right entry:

$$R(\alpha) = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} = \begin{bmatrix} \cos & -\sin \\ \sin & \cos \end{bmatrix} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}.$$

The important fact is that R's upper-right entry is "$-s$".



Second, we consider the three "basic" rotations in 3D space, i.e., the rotations around the x_1-, the x_2- and the x_3-axis. The left figure shows the rotation around the x_1-axis. The three orthonormal basis vectors $\mathbb{b}_1$, $\mathbb{b}_3$ and $\mathbb{b}_3$ — with associated axes x_1, x_2 and x_3, respectively — are rotated around the x_1-axis. Rotating $\mathbb{b}_1$, $\mathbb{b}_2$ and $\mathbb{b}_3$ by $\pi/2$ (90°) yields the orthonormal vectors $\overline{\mathbb{b}}_1$, $\overline{\mathbb{b}}_2$ and $\overline{\mathbb{b}}_3$, respectively. One obtains $\overline{\mathbb{b}}_1 = \mathbb{b}_1$, $\overline{\mathbb{b}}_2 = \mathbb{b}_3$ and $\overline{\mathbb{b}}_3 = -\mathbb{b}_2$.

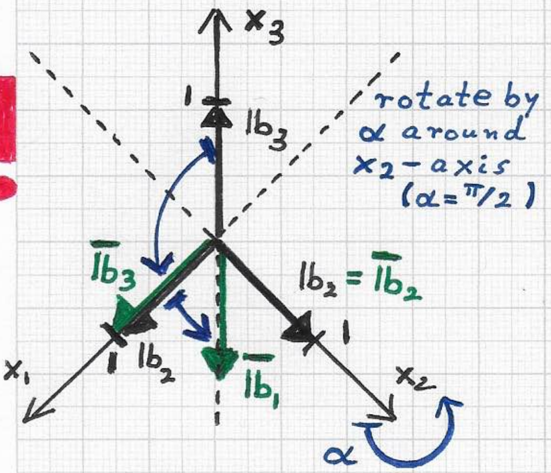
• • •

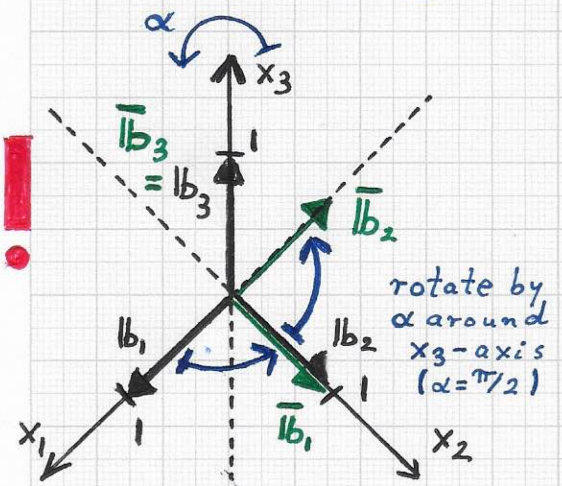
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks: ... The associated rotation matrix R_{x_1} has the following structure:

$$R_{x_1}(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{bmatrix}.$$

The important fact is that the "$-s$" entry in R_{x_1} is in the upper-right corner of the sub-matrix $\begin{bmatrix} c & -s \\ s & c \end{bmatrix}$. The basis vectors b_1, b_2 and b_3 can also be rotated around the x_2- and x_3-axis. The two left figures illustrate these two additional "basic" rotations in 3D space. Rotating b_1, b_2 and b_3 by $\pi/2$ ($90°$) around the x_2-axis yields the orthonormal vectors $\overline{b}_1 = -b_3$, $\overline{b}_2 = b_2$ and $\overline{b}_3 = b_1$; rotating b_1, b_2 and b_3 by $\pi/2$ ($90°$) around the x_3-axis yields the vectors $\overline{b}_1 = b_2$, $\overline{b}_2 = -b_1$ and $\overline{b}_3 = b_3$. We call the associated rotation matrices R_{x_2} and R_{x_3}, respectively. They have the following structures:





$$R_{x_2}(\alpha) = \begin{bmatrix} c & 0 & s \\ 0 & 1 & 0 \\ -s & 0 & c \end{bmatrix} \text{ and } R_{x_3}(\alpha) = \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Again, the important characteristic of these matrices is the placement of the "**cos**", "**sin**" and "**−sin**" entries. The underlying rule is relevant for the definition of general rotations in n-dimensional space and their associated matrices.

...