OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... Since projections of points/vectors from 4D space into the 2D plane are difficult to comprehend, we focus on the algebraic aspects of rotations in 4D space. Considering only the "basic" rotations in (x_1, x_2, x_3, x_4)-space, six rotations are combinatorially possible: There exist six possibilities to select two of the four dimensions/directions for which coordinate values remain unchanged when performing a rotation – while other coordinate values of the complementary two dimensions/directions are subjected to the "cos and sin" transformations in their associated 2D sub-space. For example, one can think of these possibilities as the following six 4-by-4 matrix structures:

$$R_{x_1x_2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c & -s \\ 0 & 0 & s & c \end{bmatrix}, \quad R_{x_1x_3} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c & 0 & -s \\ 0 & 0 & 1 & 0 \\ 0 & s & 0 & c \end{bmatrix}, \quad R_{x_1x_4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c & -s & 0 \\ 0 & s & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$R_{x_2x_3} = \begin{bmatrix} c & 0 & 0 & -s \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ s & 0 & 0 & c \end{bmatrix}, \quad R_{x_2x_4} = \begin{bmatrix} c & 0 & -s & 0 \\ 0 & 1 & 0 & 0 \\ s & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad R_{x_3x_4} = \begin{bmatrix} c & -s & 0 & 0 \\ s & c & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Notation: $R_{x_ix_j} = R_{x_ix_j}(\alpha)$; $c = \cos(\alpha)$, $s = \sin(\alpha)$

In this purely formal view of the six possibilities, the entry "$-s = -\sin(\alpha)$" is always the "upper-right entry of the sub-structure $\begin{bmatrix} c & -s \\ s & c \end{bmatrix}$." The alternate sub-structure $\begin{bmatrix} c & s \\ -s & c \end{bmatrix}$, and the respective 4-by-4 matrices $R_{x_ix_j}$, represents an orientation change of the rotation – from counter-clockwise to clock-wise – doubling the number of possibilities to 12. ...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

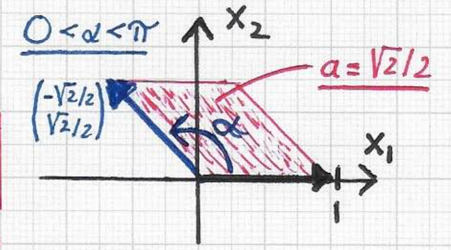
• Laplacian eigenfunctions and neural networks: …

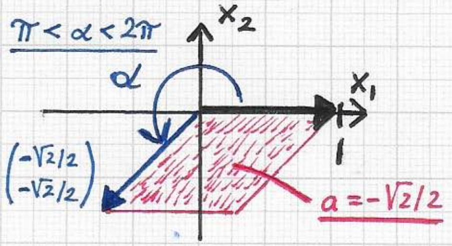
• **Note.** ∘ The orientation change of a rotation (rotating by $-\alpha$ instead of $+\alpha$) is described as follows:

$$R(-\alpha) = \begin{bmatrix} \cos(-\alpha) & -\sin(-\alpha) \\ \sin(-\alpha) & \cos(-\alpha) \end{bmatrix} = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{bmatrix}.$$

Thus, the symmetry behavior of the **cos** and **sin** functions explain the two possible sub-structures $\begin{bmatrix} c & -s \\ s & c \end{bmatrix}$ and $\begin{bmatrix} c & s \\ -s & c \end{bmatrix}$ for each rotation matrix $R_{x_i x_j}$.

∘ In this context, it is also appropriate to point out the relationship between a rotation, a signed determinant of a 2-by-2 rotation matrix and a signed area (hyper-volume) of a parallelepiped. We consider the two rotations shown in the left figures. First, we rotate the vector $(1,0)^T$ by $\frac{3}{4}\pi$, producing the vector $(-\sqrt{2}/2, \sqrt{2}/2)^T$. The value $D = \left\| \begin{bmatrix} 1 & -\sqrt{2}/2 \\ 0 & \sqrt{2}/2 \end{bmatrix} \right\| = \sqrt{2}/2$ is equal to the signed area a of the parallelepiped defined by the two vectors. Second, we rotate $(1,0)^T$

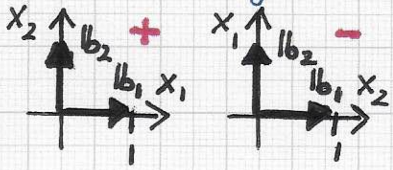




by $5/4\,\pi$, producing the vector $(-\sqrt{2}/2, -\sqrt{2}/2)^T$ and thus the signed parallelepiped area as determinant value $D = \left\| \begin{bmatrix} 1 & -\sqrt{2}/2 \\ 0 & -\sqrt{2}/2 \end{bmatrix} \right\| = -\sqrt{2}/2$. Thus, when rotating $(1,0)^T$ by an angle α, with $\pi<\alpha<2\pi$, one obtains a negative parallelepiped area, based on the value of the rotation's associated determinant. …

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

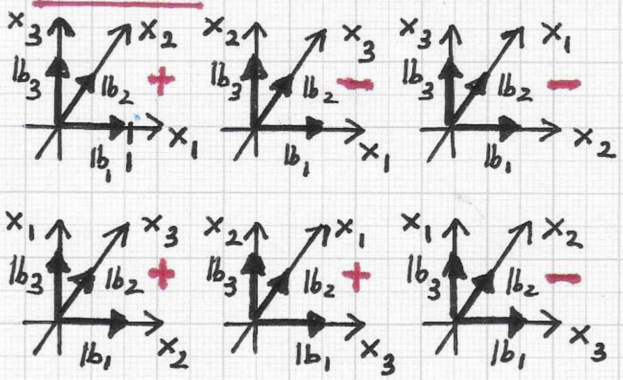
• Laplacian eigenfunctions and neural networks:... ○ Since the topic discussed here relates to determinants, signs of determinants, orientation and indexing, we briefly describe these aspects of a basis. We consider the 2D case first, shown in the left figure.



The orthonormal basis is $\{\mathbb{b}_1, \mathbb{b}_2\}$. In the left scenario, $\mathbb{b}_1 = (1,0)^T$ and $\mathbb{b}_2 = (0,1)^T$; in the right scenario, $\mathbb{b}_1 = (0,1)^T$ and $\mathbb{b}_2 = (1,0)^T$. By using the coordinate tuples of these vectors as columns of a 2-by-2 determinant, one obtains the determinant values

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \quad \text{and} \quad \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1,$$

respectively. The sign of a value can be used to call the first orientation "positive" and the second orientation "negative." The two bases are "geometrically the same," but the different indexing used indicates different basis orientations — which can be relevant for some calculations.



$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1, \quad \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -1, \quad \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1,$$

$$\begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 1, \quad \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = 1, \quad \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -1.$$

The left figure and the associated determints provided for the six possible cases for the 3D case show that three positive and three negative orientations exist. •••

■ OBJECT AND MATERIAL EIGENFUNCTIONS — Cont'd.

• **Laplacian eigenfunctions and neural networks:...** In the 3D case, one can also relate these six possibilities with the "right-hand rule" (and "left-hand rule"): The three positive orientations define right-handed $x_i x_j x_k$-coordinate systems, and the three negative orientations define left-handed $x_i x_j x_k$-coordinate systems.

For the applications we are **considering** — including the construction of an optimal orthonormal basis from a given set of non-orthogonal basis vectors — the orientation of an n-dimensional coordinate system as implied by indexing and the n system-defining basis vectors (orthonormal) is not relevant. Nevertheless, one must be aware of the fact that orientation defines the sign of determinants and that one only performs calculations producing outcomes that are invariant to orientation and determinant sign changes.

$$R_{x_1 x_2 \cdots x_{n-2}} = \begin{array}{c} 1 \\ 2 \\ \\ \\ n-1 \\ n \end{array} \begin{bmatrix} 1 & 0 & \cdots & & & 0 \\ 0 & \ddots & & & & \vdots \\ & & \ddots & & & \vdots \\ \vdots & & & 1 & 0 & 0 \\ \vdots & & & 0 & c & -s \\ 0 & \cdots & & 0 & s & c \end{bmatrix}, \ldots, \quad R_{x_3 x_4 \cdots x_n} = \begin{bmatrix} c & -s & 0 & \cdots & & 0 \\ s & c & 0 & & & \vdots \\ 0 & 0 & 1 & & & \vdots \\ \vdots & & & \ddots & & \\ \vdots & & & & \ddots & 0 \\ 0 & \cdots & & & 0 & 1 \end{bmatrix}$$

○ On page 16 (11/13/2023), the six combinatorially possible "basic" rotations $R_{x_1 x_2}, \ldots, R_{x_3 x_4}$ are provided for $n=4$. When considering the n-dimensional case, one obtains all possible rotations by determining all allowed rotations in a 2-dimensional $x_i x_j$-subspace. Two matrices are shown here.

• • •

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

- Laplacian eigenfunctions and neural networks:... One can thus determine the number of combinatorially possible "basic" rotations — from $R_{x_1 x_2 \cdots x_{n-2}}$ to $R_{x_3 x_4 \cdots x_n}$ — in the n-dimensional setting as follows: In n-dimensional space $(n \geq 2)$, a "basic" rotation is a rotation **in** one of the possible two-dimensional sub-spaces, i.e., **in** the x_1x_2-, x_1x_3-, x_1x_4-, ... x_1x_n-, x_2x_3-, ..., x_2x_n-, ..., $x_{n-1}x_n$-sub-spaces. Thus, the number of rotations is

$$(n-1)+(n-2)+\ldots+2+1 = n(n-1)/2 = (n^2-n)/2.$$

Alternatively, one can derive this number as the number of possibilities one can use to place the **cos** and **sin** terms in **two** of the available **n** rows (columns) of the n-by-n square matrix:

$$\binom{n}{2} = n!/(2!(n-2)!) = (n-1)n/2! = (n^2-n)/2.$$

Further, if one distinguishes between clockwise and counter-clockwise rotations, i.e., $\begin{pmatrix} c & -s \\ s & c \end{pmatrix}$ and $\begin{pmatrix} c & s \\ -s & c \end{pmatrix}$, the number of possible rotations is (n^2-n).

n	n^2-n
2	**2**
3	**6**
4	**12**
5	**20**
6	**30**

The left table shows the quadratically growing number of combinatorial possibilities for small n-values. In the 3D case, one achieves a "general" rotation around any axis passing through the origin by concatenating an appropriate sequence of "basic" rotations. "General" rotations around an $(n-2)$-dimensional sub-space are defined analogously for n-space...