OBJECT AND MATERIAL EIGENFUNCTIONS - Cont'd.

• Laplacian eigenfunctions and neural networks:... | • Note. (i) Quaternions and their generalization, i.e., octonions, have become desirable representations for a variety of structures and phenomena encountered in several fields of science and engineering, including: calculations of satellite/planet/aircraft/ballistic dynamics and control; physical phenomena involving space and time for their mathematical description (Maxwell's equations, for example); special and general relativity; particle physics; quantum mechenics; ...

(ii) Emmy Nöther (Noether) and, more recently, Cohl (Nicohl) Furey have substantially moved forward the science of quaternions and octonions, especially their use in mathematical physics and even the purely a priori derivation of the "Laws" governing elementary particle physics. See:

- C. Furey, "Standard Model Physics from an Algebra?", Ph. D. thesis, 2016.
- Willi Kafitz, "Hyperkomplexe Zahlen in Mathematik und Physik" (Hyper-complex numbers in mathematics and physics), 2023, University of Gießen (Giessen), Germany.
- Manon Bischoff, "From complex numbers to 'Tomb Raider'" (Von komplexen Zahlen zu 'Tomb Raider'), Spektrum der Wissenschaft, February 17, 2023.

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OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... The brief summary of Wahba's Problem, quaternions and a method for deriving a quaternion from a (rotation, orthogonal or non-orthogonal) matrix (method introduced by Itzhack Y. Bar-Itzhack) serves this purpose: Our related problem (and goal) concerns the calculation of a best orthonormal basis (or orthonormal matrix) approximation for a given set of non-orthogonal basis vectors in D-dimensional space. Wahba's Problem and the method of Bar-Itzhack (for the special case of calculating a best-approximating quaternion / 3×3 orthonormal rotation matrix for a given non-orthogonal 3×3 matrix) provide ideas and approaches that might lead to a viable solution strategy for the general D-dimensional case concerned with D×D matrices.

• Example (Bar-Itzhack method). First, we consider the simple case of reproducing an orthonormal rotation matrix from a given matrix that (already) is an orthonormal matrix. The given normalized vectors are $\mathbb{v}_1 = (1,0,0)^T$, $\mathbb{v}_2 = (0,1,0)^T$ and $\mathbb{v}_3 = (0,0,1)^T$, defining the matrices N and $\hat{N}$, see left. The eigenvalues of $\hat{N}$ are $\lambda_1 = 1$ (largest) and $\lambda_2 = \lambda_3 = \lambda_4 = -1/3$. The eigenvector $\mathbb{e}_1 = (x_0, x_1, x_2, x_3)^T$ is given by $\hat{N} \cdot \mathbb{e}_1 = 1 \cdot \mathbb{e}_1 \Rightarrow \mathbb{e}_1 = (0,0,0,x_3)^T$. ...

$$N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\hat{N} = \begin{bmatrix} -1/3 & 0 & 0 & 0 \\ 0 & -1/3 & 0 & 0 \\ 0 & 0 & -1/3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• **Laplacian eigenfunctions and neural networks:...** The value of x_3 must be chosen in order to normalize $\mathbb{e}_1$ and to define a normalized quaternion q:

$$\|\mathbb{e}_1\| = (\mathbb{e}_1 \cdot \mathbb{e}_1)^{1/2} = (x_3^{\,2})^{1/2} = 1 \Rightarrow x_3 = +/- 1 .$$

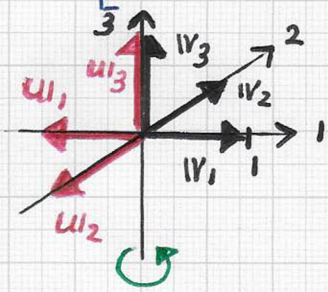
Thus, the resulting two possible quaternions are:

$$q_1 = 0 + 0i + 0j + 1k \quad , \quad q_2 = 0 + 0i + 0j - 1k .$$

(The quaternions q_1 and q_2 are conjugated normalized quaternions, i.e., $q_2 = \overline{q_1}$ and $q_1 = \overline{q_2}$.) The conversion of q_1 (or q_2) to the corresponding 3×3 matrix representation is defined on page 25 (11/25/2023); one obtains the matrix M, where

$$M = \begin{bmatrix} 1-2x_3^{\,2} & 0 & 0 \\ 0 & 1-2x_3^{\,2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} .$$

This means that the method does not reproduce the original basis vectors $\mathbb{v}_i$, $i = 1,2,3$, exactly; rather it produces a set of orthonormal vectors $\mathbb{u}_i$, $i = 1,2,3$, that one can view as the result of a rotation of the given vectors around axis '3' by rotation angle π, see figure above.



The numerical 3D example provided on pp. 3 ff. (11/3/2023) uses as non-orthogonal basis vectors $\mathbb{v}_1 = \sqrt{2}\,(2/3, 1/6, 1/6)^T$, $\mathbb{v}_2 = \sqrt{2}\,(1/6, 2/3, 1/6)^T$ and $\mathbb{v}_3 = \sqrt{2}\,(1/6, 1/6, 2/3)^T$ — for which the best-approximating vectors $\mathbb{b}_1 = (1,0,0)^T$, $\mathbb{b}_2 = (0,1,0)^T$ and $\mathbb{b}_3 = (0,0,1)^T$ are (re-)constructed. We now apply Bar-Itzhack's method to the non-orthogonal basis $\{\mathbb{v}_i\}_{i=1}^{3}$. ...

■ OBJECT AND MATERIAL EIGENFUNCTIONS — Cont'd.

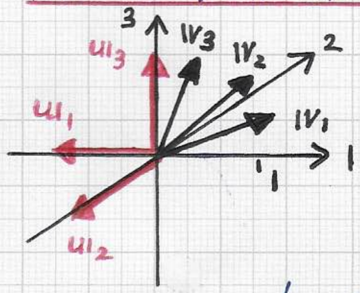
• **Laplacian eigenfunctions and neural networks:...** The resulting matrices for this set of vectors $\mathbb{v}_i$, $i = 1, 2, 3$, are:

$$\mathbb{N} = \sqrt{2} \cdot \begin{bmatrix} 2/3 & 1/6 & 1/6 \\ 1/6 & 2/3 & 1/6 \\ 1/6 & 1/6 & 2/3 \end{bmatrix}, \quad \hat{\mathbb{N}} = \begin{bmatrix} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix} \cdot \sqrt{2}/9 .$$

The eigenvalues of $\hat{\mathbb{N}}$ are $\lambda_1 = 2\sqrt{2}/3$ (largest), $\lambda_2 = \lambda_3 = -\sqrt{2}/3$ and $\lambda_4 = 0$.

Solve $\hat{\mathbb{N}} \cdot \mathbb{e}_1 = 2\sqrt{2}/3 \cdot \mathbb{e}_1$

Solving the equation system for the eigenvector $\mathbb{e}_1$ yields $\mathbb{e}_1 = (0, 0, 0, x_3)^T$. This is the same eigenvector $\mathbb{e}_1$ as obtained in the previous example.



Thus, all subsequent calculations are the same as well, and one obtains the same orthonormal matrix $\mathbb{M}$, having the orthonormal column vectors $\mathbb{u}_1$, $\mathbb{u}_2$ and $\mathbb{u}_3$ shown in the above figure together with the given non-orthogonal basis vectors $\mathbb{v}_1$, $\mathbb{v}_2$ and $\mathbb{v}_3$. ●

• **Note.** Complex numbers and their generalizations — quaternions, octonions and "hyper-complex numbers" more generally — are also becoming increasingly interesting for computational neural network applications. Thus, it is important to summarize some of the most relevant relationships between these complex numbers; rotations; orthogonal/orthonormal matrices and bases; best approximation; geometric algebra; linear and matrix algebra.

...

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• <u>Laplacian eigenfunctions and neural networks:</u>... The following list merely serves the purpose of stimulating one's thinking about related concepts and computations:

- A complex number $z = x + yi$ can be represented as a point/positional vector in the (complex) plane. $\Rightarrow$ Relationship between number and geometry.
- The multiplication of two complex numbers can be understood as and visualized by a 2D rotation.
- "Rotations in 3D space around an axis are non-commutative and require a non-commutative algebra" for their algebraic representation and calculation. Quaternions have an associated algebra that involves non-commutative multiplication.
- A rotation in 3D space around an axis passing through the origin (defined via a normalized direction vector), mapping a point p to p', can be expressed as the quaternion product $p' = q \cdot p \cdot \bar{q}$.*
- The best approximation of a given non-orthogonal 3D basis (3×3 matrix) by an orthonormal 3D basis (3×3 rotation matrix) can be elegantly computed via quaternions: A rotation-defining $q = x_0 + x_1 i + x_2 j + x_3 k$ (also defining an associated orthogonal rotation matrix), i.e., its unknown "coefficients" x_0, x_1, x_2 and x_3, can be calculated via a best approximation/least squares approach using as requirement the mapping of the non-orthogonal matrix to an orthogonal matrix. •••

* Use this equation to define the linear system for x_0, x_1, x_2, x_3, by mapping orthonormal vectors to non-orthonormal vectors.