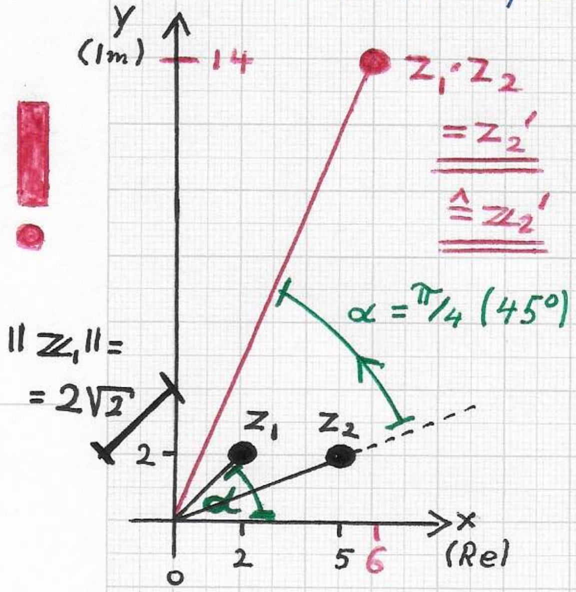
OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

- Laplacian eigenfunctions and neural networks:... We first review relevant basics of "standard" complex numbers and their associated geometrical characteristics in the (complex) plane. We consider the simple example illustrated in the left figure to explain the relationship between a rotation in the (complex) plane and the associated multiplication of complex numbers: Two complex numbers are given in the complex plane (with Re-axis x [real part] and Im-axis y [imaginary part]), z_1 and z_2.



The two numbers are $z_1 = x_1 + y_1 i = 2 + 2i$ and $z_2 = x_2 + y_2 i = 5 + 2i$. Their product is $z_1 \cdot z_2 = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1) i = 6 + 14i$, shown in the figure above. We can use the real and imaginary parts to define two corresponding points/positional vectors: $\mathbb{z}_1 = (x_1, x_2)^T = (2,2)^T$, $\mathbb{z}_2 = (x_2, y_2)^T = (5,2)^T$. Similarly, we can "map" $z_1 \cdot z_2$ to the 2D representation $\mathbb{z}_2' = (6, 14)^T = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)^T$. Using matrix multiplication, one can represent this result as $\mathbb{z}_2' = \mathbf{M} \cdot \mathbb{z}_2$:

$$\begin{pmatrix} x_1 x_2 - y_1 y_2 \\ x_1 y_2 + x_2 y_1 \end{pmatrix} = \begin{pmatrix} x_1 & -y_1 \\ y_1 & x_1 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} \|\mathbb{z}_1\| & 0 \\ 0 & \|\mathbb{z}_1\| \end{pmatrix} \cdot \begin{pmatrix} x_1/\|\mathbb{z}_1\| & -y_1/\|\mathbb{z}_1\| \\ y_1/\|\mathbb{z}_1\| & x_1/\|\mathbb{z}_1\| \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

$$= M_{scale} \cdot M_{rot} \cdot \mathbb{z}_2 = \begin{pmatrix} 2\sqrt{2} & 0 \\ 0 & 2\sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ y_2 \end{pmatrix},$$ where

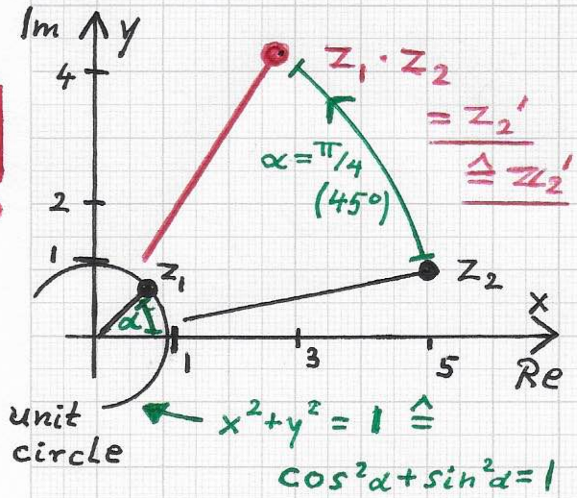
M_{scale} is a scaling matrix and M_{rot} an orthonormal matrix. ...

■ OBJECT AND MATERIAL EIGENFUNCTIONS — Cont'd.

• Laplacian eigenfunctions and neural networks:... Using the example-specific values, we obtain the mapping

$$M_{scale} \cdot M_{rot} \cdot \mathbb{z}_2 = \begin{pmatrix} 2\sqrt{2} & 0 \\ 0 & 2\sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 14 \end{pmatrix} = \mathbb{z}_2'$$

! The resulting vector $\mathbb{z}_2'$ represents the product $z_1 \cdot z_2 = 6 + 14i$. Thus, rotation in the (complex) plane and multiplication of two complex numbers are "equivalent."



• Note. The left figure shows a "pure" rotation. As we are only interested in a rotation that is "followed by a scaling with scaling factor(s) 1," we must ensure that z_1 ($\mathbb{z}_1$) is normalized, implying that $M_{scale} = \begin{pmatrix} \|\mathbb{z}_1\| & 0 \\ 0 & \|\mathbb{z}_1\| \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$.

! Thus, by requiring that $\mathbb{z}_1 = (x_1, y_1)^T$ lies on the unit circle $x^2 + y^2 = 1$, it follows that $x_1 = \cos\alpha$ and $y_1 = \sin\alpha$. Since $\|\mathbb{z}_1\| = 1$, the multiplication $z_1 \cdot z_2$ when written in equivalent matrix notation becomes $\mathbb{z}_2' = M_{rot} \cdot \mathbb{z}_2 = \begin{pmatrix} x_1 & -y_1 \\ y_1 & x_1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} c\alpha & -s\alpha \\ s\alpha & c\alpha \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

Considering the specific example shown in the figure, where $\mathbb{z}_1 = (\sqrt{2}/2, \sqrt{2}/2)^T$ and $\mathbb{z}_2 = (5, 1)^T$, one obtains $\mathbb{z}_2' = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix} \begin{pmatrix} 5 \\ 1 \end{pmatrix} = (2\sqrt{2}, 3\sqrt{2})^T$. The corresponding multiplication using complex numbers yields $z_1 \cdot z_2 = \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i\right) \cdot (5 + 1i) = 2\sqrt{2} + 3\sqrt{2}\, i$. $\Rightarrow$ $M_{rot} \cdot \mathbb{z}_2 \ \hat{=}\ z_1 \cdot z_2 = z_2'$ ...

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OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:… In summary, one can use complex number multiplication as an alternative way to represent and perform a rotation in a plane. When computing the product $z_1 \cdot z_2$, where $z_1 = x_1 + y_1 i$ and $x_1^2 + y_1^2 = 1$, the complex number z_1 acts as a "quasi-rotator," acting on z_2, and rotating $z_2 = (x_2, y_2)^T$ — the equivalent variable of $z_2 = x_2 + y_2 i$ — by the angle α; the angle α is defined by the equations $\cos\alpha = x_1$ and $\sin\alpha = y_1$. Considering our application and objective, we are not interested per se in a rotation to perform geometrical transformations. We are interested in calculating a best orthonormal basis approximation for a given basis that consists of normalized vectors not being mutually orthogonal. Therefore, we consider this question: IS IT POSSIBLE TO USE COMPLEX NUMBER MULTIPLICATION BY "MAPPING" THE ORTHONORMAL BASIS VECTORS $(1,0)^T$ AND $(0,1)^T$ SUCH THAT THE "MAPPING" (= ROTATION) RESULT IS A BEST APPROXIMATION OF A GIVEN NON-ORTHOGONAL, NORMALIZED PAIR OF BASIS VECTORS? In other words, we want to calculate the unknown "coefficients" x_1 and y_1 of a "rotator" z_1 such that z_1 defines a best approximation

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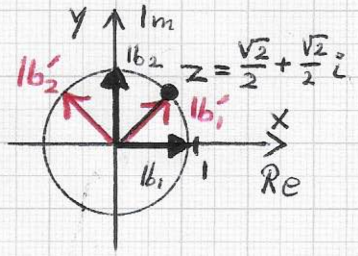
OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

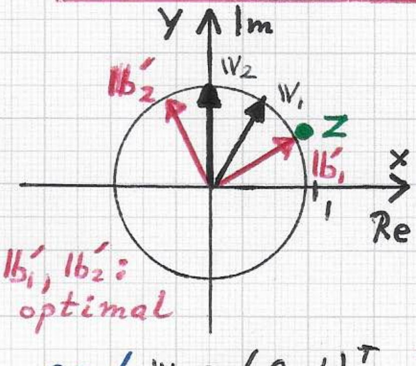
- **Laplacian eigenfunctions and neural networks:...** We first consider the simple example shown in the left figure.



The complex number $z = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$ serves as the "rotator" that implements a rotation around the origin by $\pi/4$ (45°), where $\cos\alpha = \frac{\sqrt{2}}{2}$ and $\sin\alpha = \frac{\sqrt{2}}{2}$. Thus, z is equivalent to the rotation matrix $\begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}$. We "apply" z to the standard orthonormal vectors $\mathbb{b}_1 = (1, 0)^T \mathrel{\hat{=}} 1 + 0i$ and $\mathbb{b}_2 = (0,1)^T \mathrel{\hat{=}} 0 + 1i$ by multiplying $z \cdot b_1$ and $z \cdot b_2$; we obtain the complex results

$b_1' = (\sqrt{2}/2 + \sqrt{2}/2\, i) \cdot (1 + 0i) = \sqrt{2}/2 + \sqrt{2}/2\, i \mathrel{\hat{=}} (\sqrt{2}/2, \sqrt{2}/2)^T = \mathbb{b}_1'$;

$b_2' = (\sqrt{2}/2 + \sqrt{2}/2\, i) \cdot (0 + 1i) = -\sqrt{2}/2 + \sqrt{2}/2\, i \mathrel{\hat{=}} (-\sqrt{2}/2, \sqrt{2}/2)^T = \mathbb{b}_2'$.



$\mathbb{b}_1', \mathbb{b}_2'$: optimal

We can now consider the optimization problem we want to solve. The left figure shows a numerical example. We are given two normalized, non-orthogonal basis vectors $\mathbb{w}_1 = (1/2, \sqrt{3}/2)^T$ and $\mathbb{w}_2 = (0,1)^T$. The goal is the construction of a best orthonormal approximation of the given vectors by determining the optimal rotation of the standard basis vectors $\mathbb{b}_1 = (1,0)^T$ and $\mathbb{b}_2 = (0,1)^T$ using the to-be-determined complex number $z = x + y i$, where $x^2 + y^2 = 1$, to serve as "rotator" that maps $\mathbb{b}_1$ to $\mathbb{b}_1'$ and $\mathbb{b}_2$ to $\mathbb{b}_2'$ via a least-squares minimization approach.

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■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd

• Laplacian eigenfunctions and neural networks: ... When multiplying $z = x + yi$ and the standard basis vectors one obtains $\mathbb{b}_1' = (x+yi)\cdot(1+0i) = x+yi$ and $\mathbb{b}_2' = (x+yi)\cdot(0+1i) = -y+xi$. We can now consider the corresponding vector-based squared distances between $\mathbb{v}_1$ and $\mathbb{b}_1'$ and $\mathbb{v}_2$ and $\mathbb{b}_2'$; they are:

$$d_1^2 = \| \mathbb{v}_1 - \mathbb{b}_1' \|^2 = \| (1/2, \sqrt{3}/2)^T - (x,y)^T \|^2 = (1/2 - x)^2 + (\sqrt{3}/2 - y)^2 = 1/4 - x + x^2 + 3/4 - \sqrt{3}y + y^2;$$

$$d_2^2 = \| \mathbb{v}_2 - \mathbb{b}_2' \|^2 = \| (0,1)^T - (-y,x)^T \|^2 = (0+y)^2 + (1-x)^2 = y^2 + 1 - 2x + x^2.$$

We can now define a cost function to be minimized, the sum of squared distances, i.e.,

$$D = D(x,y) = d_1^2 + d_2^2 = 2 - 3x + 2x^2 - \sqrt{3}y + 2y^2.$$

(Generally, one can understand this problem as a LAGRANGE MULTIPLIER problem, since x and y must satisfy the implicit constraint $x^2 + y^2 = 1$.)

A necessary condition for the cost function to be minimal is the requirement that $\partial/\partial x\, D = D_x = -3 + 4x = 0$ and $\partial/\partial y\, D = D_y = -\sqrt{3} + 4y = 0$. These two conditions are satisfied by the tuple $(x,y) = (3/4, \sqrt{3}/4)$. In this case, it is possible to perform normalization of this tuple at this stage: $\|(x,y)\| = (9/16 + 3/16)^{1/2} = \sqrt{3}/2$; the resulting normalized tuple is $(x,y) = (\sqrt{3}/2, 1/2)$.

Thus, the OPTIMAL "rotator" is $z = \sqrt{3}/2 + 1/2\, i$. This complex number maps b_1 to $b_1' = \sqrt{3}/2 + 1/2\, i \mathrel{\hat{=}} (\sqrt{3}/2, 1/2)^T = \mathbb{b}_1'$ and b_2 to $b_2' = -1/2 + \sqrt{3}/2\, i \mathrel{\hat{=}} (-1/2, \sqrt{3}/2)^T = \mathbb{b}_2'$. ⇒ OPTIMAL RESULT!

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