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▪ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks: ... Rotation in 3D space around a directed axis passing through the origin can be represented via standard matrix algebra or via quaternion algebra — two equivalent ways to describe rotation. For the purpose of of this discussion, we consider the scenario of rotation of a point $\mathbf{p}=(x,y,z)^T$ around an axis defined by the normalized direction vector $\mathbf{a}=(a_1,a_2,a_3)^T$, $\|\mathbf{a}\|=1$, using a rotation angle α.

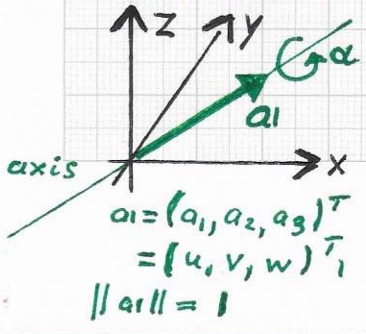
This rotation can be expressed by the matrix R, given as:

$$R=\begin{bmatrix} a_1^2+(1-a_1^2)c & a_1a_2(1-c)-a_3s & a_1a_3(1-c)+a_2s \\ a_2a_1(1-c)+a_3s & a_2^2+(1-a_2^2)c & a_2a_3(1-c)-a_1s \\ a_3a_1(1-c)-a_2s & a_3a_2(1-c)+a_1s & a_3^2+(1-a_3^2)c \end{bmatrix},$$

where $s=\sin=\sin(\alpha)$ and $c=\cos=\cos(\alpha)$. We can further simplify the notation by writing $\mathbf{a}=(u,v,w)^T$:

$$R=\begin{bmatrix} u^2(1-c)+c & uv(1-c)-ws & uw(1-c)+vs \\ vu(1-c)+ws & v^2(1-c)+c & vw(1-c)-us \\ wu(1-c)-vs & wv(1-c)+us & w^2(1-c)+c \end{bmatrix}.$$

The matrix R has an obvious combinatorial structure. Since R is an orthonormal rotation matrix, the inverse matrix is $R^{-1}=R^T$. This rotation matrix can be derived in several ways. For example, "Computational Geometry for Design and Manufacture," I.D. Faux and M.J. Pratt, Ellis Hoorwood, 1979, is a good reference.



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■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks: ... Thus, in matrix notation a point $\mathbb{p} = (x, y, z)^T$ is mapped to its new position $\mathbb{p}' = (x', y', z')^T$ by calculating $\mathbb{p}' = R \cdot \mathbb{p}$. On page 23 (11/24/2023), the essential rules for the multiplication of two quaternions are provided. It is possible to represent the considered rotation via quaternions in an equivalent way. Again, one can derive the quaternion-based representation in several ways. Here, we are merely interested in the algebraic definition of the rotation. First, performing a rotation by α is expressed as "performing two times a rotation by $\alpha/2$." Second, the point to be rotated has the representation $p = 0 + xi + yj + zk$. Third, the coefficients/components of the rotation axis direction vector $\mathbb{a} = (a_1, a_2, a_3)^T = (u, v, w)^T$ become the coefficients for the three complex terms of a quaternion, i.e., $ui + vj + wk$. Fourth, the "hyper-complex" number (quaternion) p is mapped to its rotated image by using two "rotator quaternions," i.e., $q = \hat{c} + (ui + vj + wk)\hat{s}$ and $\bar{q} = \hat{c} - (ui + vj + wk)\hat{s}$. Here, one defines $\hat{c} = \cos(\alpha/2)$ and $\hat{s} = \sin(\alpha/2)$. Finally, one can compute p' as $p' = q \cdot p \cdot \bar{q}$.

• **Note.** It is important to realize that the multiplication of quaternions is ASSOCIATIVE: $p' = (q \cdot p) \cdot \bar{q}$
$= q \cdot (p \cdot \bar{q})$.

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■ OBJECT AND MATERIAL EIGENFUNCTIONS — Cont'd.

- Laplacian eigenfunctions and neural networks: … We can demonstrate that the formulae $\mathbf{x}' = R \cdot \mathbf{x}$ and $p' = q \cdot p \cdot \bar{q}$ are equivalent ways to represent the rotation of a point $\mathbf{x} = (x, y, z)^T \mathrel{\hat{=}} p = 0 + xi + yj + zk$ by the rotation angle α around the directed axis defined by the normalized direction vector $(u, v, w)^T$ and passing through the origin. We must calculate $\mathbf{x}'$ and p' and compare the resulting expressions for x', y' and z' to show equivalence. First, we compute $p' = q \cdot (p \cdot \bar{q})$:

$$
\begin{aligned}
p' = {} & (\hat{c} + u\hat{s}i + v\hat{s}j + w\hat{s}k) \cdot \\
& \cdot \big((0 + xi + yj + zk) \cdot (\hat{c} - u\hat{s}i - v\hat{s}j - w\hat{s}k) \big) \\
= {} & (\hat{c} + u\hat{s}i + v\hat{s}j + w\hat{s}k) \cdot \\
& \big((0 + xu\hat{s} + yv\hat{s} + zw\hat{s}) + (0 + x\hat{c} - yw\hat{s} + zv\hat{s})\, i \\
& \quad + (0 + xw\hat{s} + y\hat{c} - zu\hat{s})\, j + (0 - xv\hat{s} + yu\hat{s} + z\hat{c})\, k \big) \\
= {} & \big(\hat{c}(xu\hat{s} + yv\hat{s} + zw\hat{s}) - u\hat{s}(x\hat{c} - yw\hat{s} + zvs) \\
& \qquad - v\hat{s}(xw\hat{s} + y\hat{c} - zu\hat{s}) - w\hat{s}(-xv\hat{s} + yu\hat{s} + z\hat{c}) \big) \\
& + \big(\hat{c}(x\hat{c} - yw\hat{s} + zv\hat{s}) + u\hat{s}(xu\hat{s} + yv\hat{s} + zw\hat{s}) \\
& \qquad + v\hat{s}(-xv\hat{s} + yu\hat{s} + z\hat{c}) - w\hat{s}(xw\hat{s} + y\hat{c} - zu\hat{s}) \big)\, i \\
& + \big(\hat{c}(xw\hat{s} + y\hat{c} - zu\hat{s}) - u\hat{s}(-xv\hat{s} + yu\hat{s} + z\hat{c}) \\
& \qquad + v\hat{s}(xu\hat{s} + yv\hat{s} + zw\hat{s}) + w\hat{s}(x\hat{c} - yw\hat{s} + zv\hat{s}) \big)\, j \\
& + \big(\hat{c}(-xv\hat{s} + yu\hat{s} + z\hat{c}) + u\hat{s}(xw\hat{s} + y\hat{c} - zu\hat{s}) \\
& \qquad - v\hat{s}(x\hat{c} - yw\hat{s} + zv\hat{s}) + w\hat{s}(xu\hat{s} + yv\hat{s} + zw\hat{s}) \big)\, k \\
= {} & \big(\hat{s}\hat{c}(xu + yv + zw) - \hat{s}\hat{c}xu - u\hat{s}^2(-yw + zv) - \hat{s}\hat{c}yv - v\hat{s}^2(xw - zu) \\
& \qquad - \hat{s}\hat{c}zw - w\hat{s}^2(-xv + yu) \big) \\
& + \ldots
\end{aligned}
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■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks: ...

$$+\left(\hat{c}^2 x + \hat{s}\hat{c}(-yw+zv) + u\hat{s}^2(xu+yv+zw) + v\hat{s}^2(-xv+yu) + \hat{s}\hat{c}\,zv - w\hat{s}^2(xw-zu) - \hat{s}\hat{c}\,yw\right) i$$

$$+\left(\hat{c}^2 y + \hat{s}\hat{c}(xw-zu) - u\hat{s}^2(-xv+yu) - \hat{s}\hat{c}\,zu + v\hat{s}^2(xu+yv+zw) + w\hat{s}^2(-yw+zv) + \hat{s}\hat{c}\,xw\right) j$$

$$+\left(\hat{c}^2 z + \hat{s}\hat{c}(-xv+yu) + u\hat{s}^2(xw-zu) + \hat{s}\hat{c}\,yu - v\hat{s}^2(-yw+zv) - \hat{s}\hat{c}\,xv + w\hat{s}^2(xu+yv+zw)\right) k$$

$$= \left(\hat{s}^2\left(-u(-yw+zv) - v(xw-zu) - w(-xv+yu)\right)\right)$$

$$+\left(\hat{c}^2 x + \hat{s}\hat{c}(-yw+zv+zv-yw) + \hat{s}^2\left(u(xu+yv+zw) + v(-xv+yu) - w(xw-zu)\right)\right) i$$

$$+\left(\hat{c}^2 y + \hat{s}\hat{c}(xw-zu-zu+xw) + \hat{s}^2\left(-u(-xv+yu) + v(xu+yv+zw) + w(-yw+zv)\right)\right) j$$

$$+\left(\hat{c}^2 z + \hat{s}\hat{c}(-xv+yu+yu-xv) + \hat{s}^2\left(u(xw-zu) - v(-yw+zv) + w(xu+yv+zw)\right)\right) k$$

$$= \left(\hat{s}^2(yuw - zuv - xvw + zuv + xvw - yuw)\right)$$

$$+\left(\hat{c}^2 x + 2\hat{s}\hat{c}(zv-yw) + \hat{s}^2(xu^2 + yuv + zuw - xv^2 + yuv - xw^2 + zuw)\right) i$$

$$+\left(\hat{c}^2 y + 2\hat{s}\hat{c}(xw-zu) + \hat{s}^2(xuv - yu^2 + xuv + yv^2 + zvw - yw^2 + zvw)\right) j$$

$$+\left(\hat{c}^2 z + 2\hat{s}\hat{c}(yu-xv) + \hat{s}^2(xuw - zu^2 + yvw - zv^2 + xuw + yvw + zw^2)\right) k$$

$$= 0 + \left(\hat{c}^2 x + 2\hat{s}\hat{c}(zv-yw) + \hat{s}^2\left(x(u^2-v^2-w^2) + 2u(yv+zw)\right)\right) i$$

$$+\left(\hat{c}^2 y + 2\hat{s}\hat{c}(xw-zu) + \hat{s}^2\left(y(-u^2+v^2-w^2) + 2v(xu+zw)\right)\right) j$$

$$+\left(\hat{c}^2 z + 2\hat{s}\hat{c}(yu-xv) + \hat{s}^2\left(z(-u^2-v^2+w^2) + 2w(xu+yv)\right)\right) k$$

! { Use "half-angle formulae":

i) $\sin(\alpha/2) = \mathrm{sgn}(\sin(\alpha/2)) \cdot ((1-\cos\alpha)/2)^{1/2} = \hat{s}$

$\Rightarrow \sin^2(\alpha/2) = (1-\cos\alpha)/2 = (1-c)/2$

ii) $\cos(\alpha/2) = \mathrm{sgn}(\cos(\alpha/2)) \cdot ((1+\cos\alpha)/2)^{1/2} = \hat{c}$

$\Rightarrow \cos^2(\alpha/2) = (1+\cos\alpha)/2 = (1+c)/2$

iii) $\sin(\alpha/2)\cdot\cos(\alpha/2) = \mathrm{sgn}(\sin(\alpha/2)) \cdot ((1-\cos\alpha)/2)^{1/2} \cdot \mathrm{sgn}(\cos(\alpha/2)) \cdot ((1+\cos\alpha)/2)^{1/2}$

$= SGN \cdot (1-c^2)^{1/2}/2$ }

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OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:...

$$= 0 + \left(\frac{x}{2}(1+c) + 2(zv - yw)\cdot SGN \cdot (1-c^2)^{1/2}/2 + \left(x(u^2-v^2-w^2) + 2u(yv+zw) \right) \frac{1-c}{2} \right) i$$

$$+ \left(\frac{y}{2}(1+c) + 2(xw - zu)\cdot SGN \cdot (1-c^2)^{1/2}/2 + \left(y(-u^2+v^2-w^2) + 2v(xu+zw) \right) \frac{1-c}{2} \right) j$$

$$+ \left(\frac{z}{2}(1+c) + 2(yu - xv)\cdot SGN \cdot (1-c^2)^{1/2}/2 + \left(z(-u^2-v^2+w^2) + 2w(xu+yv) \right) \frac{1-c}{2} \right) k$$

{ with $SGN = +1$ or $SGN = -1$ }

$$= 0 + \left(\frac{x}{2}(1+c) + (zv - yw)\cdot SGN \cdot s + \left(\frac{x}{2}(u^2-v^2-w^2) + u(yv+zw) \right)(1-c) \right) i$$

$$+ \left(\frac{y}{2}(1+c) + (xw - zu)\cdot SGN \cdot s + \left(\frac{y}{2}(-u^2+v^2-w^2) + v(xu+zw) \right)(1-c) \right) j$$

$$+ \left(\frac{z}{2}(1+c) + (yu - xv)\cdot SGN \cdot s + \left(\frac{z}{2}(-u^2-v^2+w^2) + w(xu+yv) \right)(1-c) \right) k$$

{ where $s = (1-c^2)^{1/2}$ since $s^2 + c^2 = 1$ }

{ product rule: $\sin(\alpha)\cdot\cos(\alpha)$

$$= (\sin(\alpha+\alpha) + \sin(\alpha-\alpha))/2$$

$$= (\sin(2\alpha))/2$$

$$\Rightarrow \hat{s}\hat{c} = (\sin(\alpha))/2 = s/2$$

$$\Rightarrow SGN = +1$$ }

$= 0 + x'i + y'j + z'k$, where:

$$x' = \frac{x}{2}(1+c) + (zv - yw)s + \left(\frac{x}{2}(u^2-v^2-w^2) + u(yv+zw) \right)(1-c)$$

$$= \left(\tfrac{1}{2}(1+c) + \tfrac{1}{2}(1-c)(u^2-v^2-w^2) \right) \cdot x$$

$$+ \left(-ws + uv(1-c) \right) \cdot y$$

$$+ \left(vs + uw(1-c) \right) \cdot z \;;$$

$$y' = \ldots \left(\tfrac{1}{2}(1+c) + \tfrac{1}{2}(1-c)(-u^2+v^2-w^2) \right) \cdot y$$

$$+ \left(ws + uv(1-c) \right) \cdot x$$

$$+ \left(-us + vw(1-c) \right) \cdot z \;;$$

$$z' = \ldots \left(\tfrac{1}{2}(1+c) + \tfrac{1}{2}(1-c)(-u^2-v^2+w^2) \right) \cdot z$$

$$+ \left(-vs + uw(1-c) \right) \cdot x$$

$$+ \left(us + vw(1-c) \right) \cdot y \;.$$

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