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12/11/2023

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... This detailed example demonstrates how one can use quaternions to represent and execute rotations elegantly in 3D space. Further, the example shows that a quaternion-based optimization approach for performing rotation of an orthonormal 3D basis to optimally approximate a given non-orthogonal set of normalized vectors is effective and computationally efficient; the determination of the approximation-error-minimizing optimal rotation angle is defined by a univariate function minimization. Unfortunately, the generalization of quaternions to higher-dimensional spaces is complicated and is subject to ongoing research in algebra, geometric algebra and mathematical physics. Generally, the resulting "number systems" lose properties with increasing dimension (loss of commutative and associative properties, for example). Such "hyper-complex" number systems involve one, three, seven or 15 imaginary parts – called complex numbers, quaternions, octonions and sedenions, respectively. The above table is the multiplication table for quaternions, involving the imaginary parts i_1, i_2 and i_3. ...

$\cdot$	1	i_1	i_2	i_3
1	1	i_1	i_2	i_3
i_1	i_1	-1	i_3	$-i_2$
i_2	i_2	$-i_3$	-1	i_1
i_3	i_3	i_2	$-i_1$	-1

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... Multiplication table for complex numbers, quaternions, octonions and sedenions:

•	1	i_1	i_2	i_3	i_4	i_5	i_6	i_7	i_8	i_9	i_{10}	i_{11}	i_{12}	i_{13}	i_{14}	i_{15}
1	1	i_1	i_2	i_3	i_4	i_5	i_6	i_7	i_8	i_9	i_{10}	i_{11}	i_{12}	i_{13}	i_{14}	i_{15}
i_1	i_1	-1	i_3	$-i_2$	i_5	$-i_4$	$-i_7$	i_6	i_9	$-i_8$	$-i_{11}$	i_{10}	$-i_{13}$	i_{12}	i_{15}	$-i_{14}$
i_2	i_2	$-i_3$	-1	i_1	i_6	i_7	$-i_4$	$-i_5$	i_{10}	i_{11}	$-i_8$	$-i_9$	$-i_{14}$	$-i_{15}$	i_{12}	i_{13}
i_3	i_3	i_2	$-i_1$	-1	i_7	$-i_6$	i_5	$-i_4$	i_{11}	$-i_{10}$	i_9	$-i_8$	$-i_{15}$	i_{14}	$-i_{13}$	i_{12}
i_4	i_4	$-i_5$	$-i_6$	$-i_7$	-1	i_1	i_2	i_3	i_{12}	i_{13}	i_{14}	i_{15}	$-i_8$	$-i_9$	$-i_{10}$	$-i_{11}$
i_5	i_5	i_4	$-i_7$	i_6	$-i_1$	-1	$-i_3$	i_2	i_{13}	$-i_{12}$	i_{15}	$-i_{14}$	i_9	$-i_8$	i_{11}	$-i_{10}$
i_6	i_6	i_7	i_4	$-i_5$	$-i_2$	i_3	-1	$-i_1$	i_{14}	$-i_{15}$	$-i_{12}$	i_{13}	i_{10}	$-i_{11}$	$-i_8$	i_9
i_7	i_7	$-i_6$	i_5	i_4	$-i_3$	$-i_2$	i_1	-1	i_{15}	i_{14}	$-i_{13}$	$-i_{12}$	i_{11}	i_{10}	$-i_9$	$-i_8$
i_8	i_8	$-i_9$	$-i_{10}$	$-i_{11}$	$-i_{12}$	$-i_{13}$	$-i_{14}$	$-i_{15}$	-1	i_1	i_2	i_3	i_4	i_5	i_6	i_7
i_9	i_9	i_8	$-i_{11}$	i_{10}	$-i_{13}$	i_{12}	i_{15}	$-i_{14}$	$-i_1$	-1	$-i_3$	i_2	$-i_5$	i_4	i_7	$-i_6$
i_{10}	i_{10}	i_{11}	i_8	$-i_9$	$-i_{14}$	$-i_{15}$	i_{12}	i_{13}	$-i_2$	i_3	-1	$-i_1$	$-i_6$	$-i_7$	i_4	i_5
i_{11}	i_{11}	$-i_{10}$	i_9	i_8	$-i_{15}$	i_{14}	$-i_{13}$	i_{12}	$-i_3$	$-i_2$	i_1	-1	$-i_7$	i_6	$-i_5$	i_4
i_{12}	i_{12}	i_{13}	i_{14}	i_{15}	i_8	$-i_9$	$-i_{10}$	$-i_{11}$	$-i_4$	i_5	i_6	i_7	-1	$-i_1$	$-i_2$	$-i_3$
i_{13}	i_{13}	$-i_{12}$	i_{15}	$-i_{14}$	i_9	i_8	i_{11}	$-i_{10}$	$-i_5$	$-i_4$	i_7	$-i_6$	i_1	-1	i_3	$-i_2$
i_{14}	i_{14}	$-i_{15}$	$-i_{12}$	i_{18}	i_{10}	$-i_{11}$	i_8	i_9	$-i_6$	$-i_7$	$-i_4$	i_5	i_2	$-i_3$	-1	i_1
i_{15}	i_{15}	i_{14}	$-i_{13}$	$-i_{12}$	i_{11}	i_{10}	$-i_9$	i_8	$-i_7$	i_6	$-i_5$	$-i_4$	i_3	i_2	$-i_1$	-1

• **Note.** The 16×16 multiplication table for sedenions includes the 2×2, 4×4 and 8×8 multiplication tables for complex numbers, quaternions and octonions as "sub-tables" in the upper-left corner, as indicated. More recently, octonions and sedenions have gained significant interest in STRING theory.

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OBJECT AND MATERIAL EIGENFUNCTIONS - Cont'd.

• Laplacian eigenfunctions and neural networks:... • Note. Octonions can be used to represent rotations in an 8-dimensional space. Since octonion multiplication is not associative, calculations become more complicated. While this topic is interesting, its discussion goes beyond the scope of these notes. The literature covering octonions ("Cayley numbers"), Cayley algebra and Clifford algebra provides details.

It turns out that quaternions have also become relevant for computational neural network problems and applications. Quaternions gained substantial interest in computer graphics, especially computer animation, since they provide an elegant and computationally efficient and (in some sense) numerically stable tool for performing "smooth rotations." While computer animation employs quaternions to represent and perform rotations as part of "designing" an object's or character's movements, quaternions can also be used for devising features/characterictics of objects in images, videos or 3D volumetric scans. Thus, quaternions are briefly reviewed at this point with a focus on their use in animation and data analysis (for object description and recognition).

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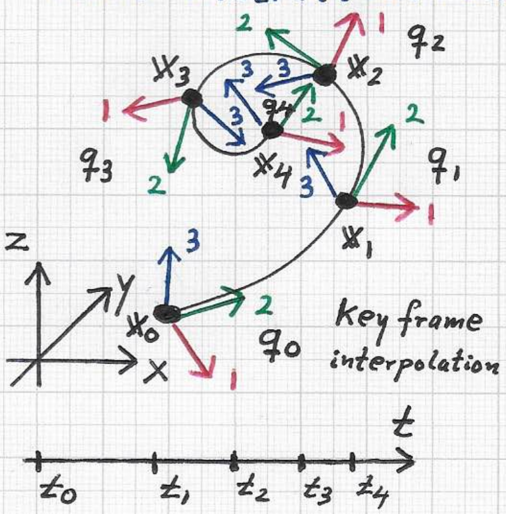
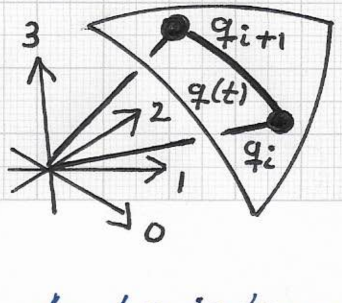
■ OBJECT AND MATERIAL EIGENFUNCTIONS — Cont'd.

• Laplacian eigenfunctions and neural networks:... Defining and smoothly interpolating "key frames" is crucially important in computer animation. More precisely, one must typically specify an object's position and orientation for parameter values (time points) $t_0, \dots, t_n$ and subsequently interpolate the specified data — defining the desired spline over the interval $[t_0, t_n]$ that can subsequently be evaluated at the necessary time discretization/sampling for actual rendering.



The left figure illustrates the basic problem: Given are time points $t_0, \dots, t_4$ with associated positions $x_0, \dots, x_4$ in 3D space and corresponding orientations — called $q_0, \dots, q_4$ in the figure. The goal is the computation of an interpolant that defines a smoothly interpolated position $x(t)$ with orientation $q(t)$ for $t \in [t_0, t_4]$. This interpolant is constructed in "4D quaternion space." The left figure shows the two unit quaternions q_i and q_{i+1} — resulting by mapping two keyframe rotation/orientation matrices to q_i and q_{i+1} — to be interpolated by $q(t)$ that lies on the unit 4D hyper-sphere.



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OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks: ... Thus, instead of interpolating the specified keyframe rotation matrices in 3D space, the matrices are mapped to their "unit quaternion representation equivalents"; one can then compute an interpolating "quaternion curve" in 4D space, subject to satisfying that the final interpolating "quaternion curve" $q(t)$ consists only of unit quaternions that properly represent a rotation matrix for 3D space, for every $t \in [t_0, t_n]$. One can consider a multitude of interpolation methods for the construction of a continuous and smooth curve $q(t)$. (For example, the paper "υ-Quaternion Splines for the Smooth Interpolation of Orientations" by Gregory M. Nielson, IEEE Trans. on Visualization and Computer Graphics 10(2), 2004, pp. 224-229, describes an elegant scheme based on the concepts of B-splines and cubic splines using a tension parameter (υ).) One might also consider the use of an interpolation method that first interpolates the quaternions $q_0, \dots, q_n$ smoothly via a curve $q(t)$ that is not forced to lie on the unit 4D hyper-sphere (not forced to define exclusively unit quaternions $q(t)$ for all t-values). A resulting $q(t)$ curve point can be normalized in a subsequent step, thereby ensuring that $q(t)/\|q(t)\|$ defines a proper rotation matrix.

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