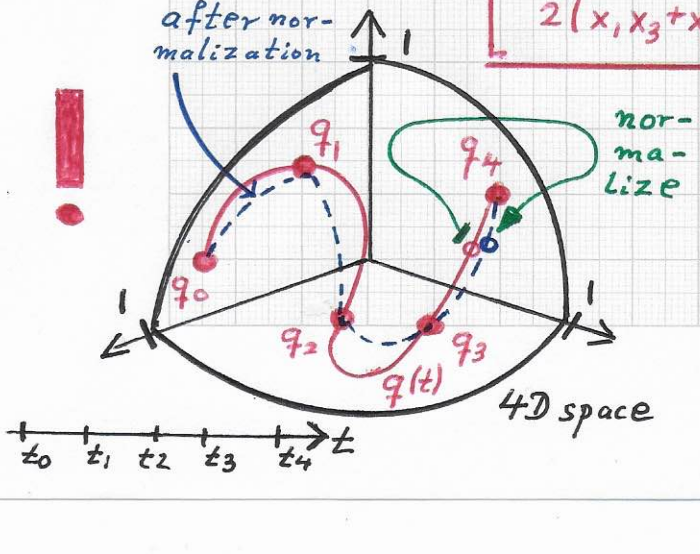
▪ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... It is possible to map a 3×3 rotation matrix M directly to its corresponding quaternion and vice versa. A 3×3 rotation matrix M satisfies $MM^t = I$, i.e., $M^{-1} = M^t$, and $|M| = \det M = +1$. The matrix M defines a rotation around the axis passing through the origin with unit direction vector $u = (u_1, u_2, u_3)^T$, where this vector is associated with M's unit eigenvalue. Further, the rotation angle α is given as the value defined by $1 + 2\cos\alpha = m_{11} + m_{22} + m_{33} = \text{trace } M$, i.e., $\cos\alpha = (m_{11} + m_{22} + m_{33} - 1)/2$. We write a quaternion again as $q = x_0 + x_1 i + x_2 j + x_3 k$. Here, we consider only normalized quaternions for which $\|q\|^2 = \sum_{i=0}^{3} x_i^2 = 1$. The resulting mappings are:

- $q(M) = x_0 + x_1 i + x_2 j + x_3 k$
  $= \cos\alpha/2 + u_1 \sin\alpha/2 \; i + u_2 \sin\alpha/2 \; j + u_3 \sin\alpha/2 \; k$.

- $$M(q) = \begin{bmatrix} 1-2(x_2^2+x_3^2) & 2(x_1x_2+x_0x_3) & 2(x_1x_3-x_0x_2) \\ 2(x_1x_2-x_0x_3) & 1-2(x_1^2+x_3^2) & 2(x_2x_3+x_0x_1) \\ 2(x_1x_3+x_0x_2) & 2(x_2x_3-x_0x_1) & 1-2(x_1^2+x_2^2) \end{bmatrix}.$$



The left figure shows how one can construct an interpolating curve $q(t)$ for a given set of unit quaternions q_i. Since $q(t)$, $t_0 \le t \le t_4$, is generally not a unit quaternion, one must normalize it. ...

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

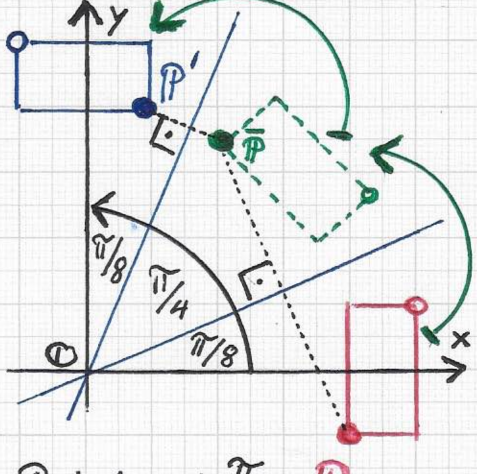
• Laplacian eigenfunctions and neural networks:... In summary, the possibility to map an orthonormal rotation matrix M to an equivalent unit quaternion in "4D quaternion space," $q(M)$, allows one to construct an interpolating curve $q(t)$ for a set of given quaternions $q_i = q(t_i) = q(M(t_i))$ — where original rotation matrices $M_i = M(t_i)$ are mapped to q_i, $i = 0, \ldots, n$. As one can view a quaternion $q = x_0 + x_1 i + x_2 j + x_3 k$ as a "point" $(x_0, x_1, x_2, x_3)^T$, one can employ a (curve) interpolation scheme to perform point-interpolation in 4D (x_0, x_1, x_2, x_3)-space. While the given quaternions q_i, $i = 0, \ldots, n$, are unit quaternions, a resulting curve point $q(t)$, $t_0 \leq t \leq t_n$, does not lie on the 3-manifold hyper-sphere in 4D space defined as $\|q\|^2 = \sum_{i=0}^{3} x_i^2 = 1$. Thus, one must normalize $q(t)$ to force it to lie on the unit hyper-sphere: $q(t) := q(t)/\|q(t)\|$. By ensuring that normalized $q(t)$ curve points lie on the unit hyper-sphere, all quaternions define proper rotation matrices $M = M(q) = M(q(t))$. One can consider various interpolation methods to define the continuous curve $q(t)$, e.g., linear, cubic or tension splines. IT IS POSSIBLE TO USE INTERPOLATION METHODS THAT PRODUCE A CURVE $q(t)$ LYING ENTIRELY ON THE HYPER-SPHERE $\|q\| = 1$.

...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks: …

• **Notes concerning rotation and reflection.** Recently, quaternions and also octonions have also been used for the design and implementation of computational neural networks. In the context of object recognition and classification, scientists have reported that the use of quaternions (and to some degree also octonions) can improve both recognition results and computational efficiency. One important geometrical aspect closely related to this topic is the fact that one can perform a rotation purely by the execution of reflections (The geometric algebra literature discusses this "equivalence" of rotation and reflection in detail.) We briefly discuss the case of rotation and reflection in the plane before describing the situation in 3D space.



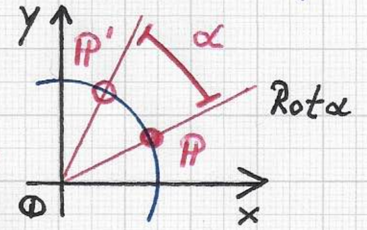
Rotation by $\pi/2$. $\mathbb{P}$

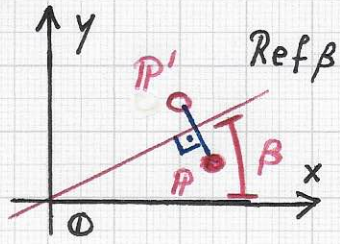
"Two reflections used to perform a rotation."

The left figure provides a simple example. Here, a rotation by the angle $\alpha = \pi/2$ is achieved via two reflections using as reflection lines the lines that pass through the origin and have angles relative to the x-axis of $\pi/8$ and $3\pi/8$, respectively. The first reflection maps $\mathbb{P}$ to $\overline{\mathbb{P}}$; the second maps $\overline{\mathbb{P}}$ to the designation $\mathbb{P}'$. …

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• **Laplacian eigenfunctions and neural networks:...** We now discuss the general case for rotations and reflections in the 2D plane. We call a rotation matrix of a rotation around the origin by angle α $Rot\alpha$; we call a reflection matrix of a reflection with respect to a line passing through the origin and defining an angle β relative to the x-axis $Ref\beta$. The left figure shows these transformations, mapping a point $\mathbb{P}$ to $\mathbb{P}'$. The two matrices are:

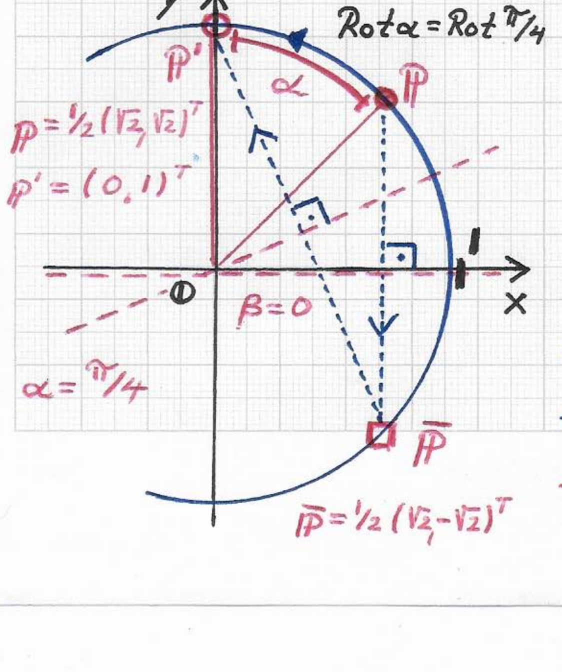




$$Rot\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \text{ and } Ref\beta = \begin{bmatrix} \cos 2\beta & \sin 2\beta \\ \sin 2\beta & -\cos 2\beta \end{bmatrix}.$$

(Note. $\det(Rot\alpha) = 1$ and $\det(Ref\beta) = -1$.)

In order to represent a rotation as the concatenation of two reflections, one can use one/some of the facts that apply to four possible matrix products:



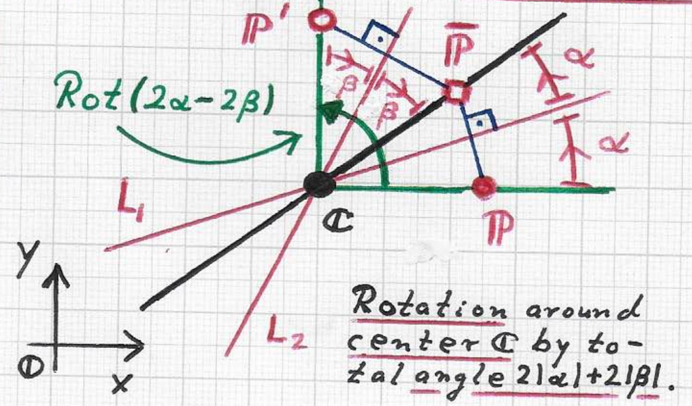
$$\boxed{\begin{aligned} Rot\alpha\; Rot\beta &= Rot(\alpha+\beta) \\ Ref\alpha\; Ref\beta &= Rot(2(\alpha-\beta)) \\ Rot\alpha\; Ref\beta &= Ref(\alpha/2+\beta) \\ Ref\alpha\; Rot\beta &= Ref(\alpha-\beta/2) \end{aligned}}$$

For example, the second equation yields for the configuration in the left figure:

$$Rot\pi/4 = Rot(2(\pi/8 - 0)) = Ref\,\pi/8\; Ref\,0 =$$
$$= \begin{bmatrix} \cos\pi/4 & \sin\pi/4 \\ \sin\pi/4 & -\cos\pi/4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}. \ldots$$

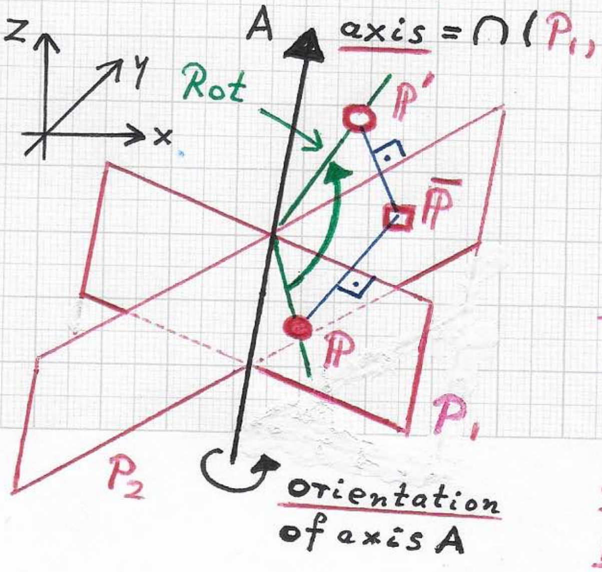
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• **Laplacian eigenfunctions and neural networks:...** In the following, we discuss the geometrial aspects of the fundamental "theorem" $Rot(2(\alpha-\beta)) = Ref\alpha \cdot Ref\beta$ for rotation in 2D and 3D spaces. The general 2D case illustrated in the left figure considers a rotation in the 2D plane around an arbitrary center $\mathbb{C}$. The two lines used for reflections, L_1 and L_2, intersect in the center of rotation, $\mathbb{C}$. The (smaller) angle between L_1 and L_2 has the value $|\alpha|+|\beta|$. The L_1-reflection of $\mathbb{P}$ yields $\bar{\mathbb{P}}$, and the subsequent L_2-reflection of $\bar{\mathbb{P}}$ yields $\mathbb{P}'$. Considering the angles (and their signs defined by orientation) shown, $\mathbb{P}$ is mapped to $\mathbb{P}'$ via the rotation matrix $Rot(2\alpha-2\beta) = Rot(2(\alpha-\beta))$.



Rotation around center $\mathbb{C}$ by total angle $2|\alpha|+2|\beta|$.

The left figure shows the 3D case. Rotation is done with respect to an arbitrary oriented axis, which is the intersection line of the two reflection planes, P_1 and P_2. P_1-reflection of $\mathbb{P}$ yields $\bar{\mathbb{P}}$, P_2-reflection of $\bar{\mathbb{P}}$ yields $\mathbb{P}'$. THUS, "$Rot_A = Ref_{P_2} \cdot Ref_{P_1}$"!



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