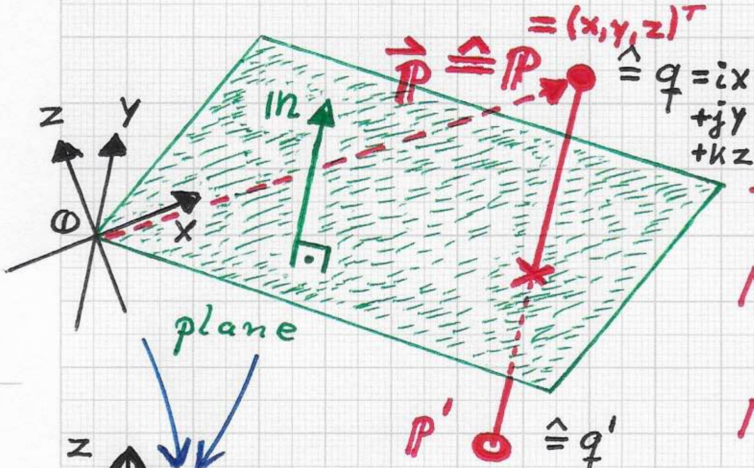
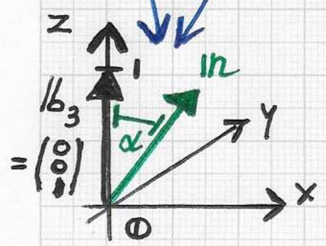
OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks: ...

• **Reflection** — with respect to a plane passing through the origin and having "outward" unit normal vector $\mathbb{n} = (n_x, n_y, n_z)^T$, $\|\mathbb{n}\| = 1$. This transformation is illustrated in the left figure.





$\mathbb{P}' = (x', y', z')^T = M_3 \cdot M_2 \cdot M_1 \cdot \mathbb{P}$
$= M_3 \cdot M_2 \cdot M_1 \cdot (x, y, z)^T$, where

M_1 = matrixMappingVector_n ToVector_001 ,

M_2 = matrixReflectingWith RespectToXYplane ,

$M_3 = M^{-1}$ (= inverse of matrix M_1)

The left figure shows the relevent data needed for the definition of M_1, i.e., $\mathbb{n}$, α and $\mathbb{b}_3$.

In order to express this reflection in quaternion notation, we understand the point $\mathbb{P}$ as positional vector $\vec{\mathbb{P}}$, see figure above. Thus, we can represent all x-, y- and z-coordinate values of $\mathbb{P}$ and $\mathbb{n}$ via the corresponding **complex** i-, j- and k-terms of the equivalent quaternions. In the following, we will show that this reflection can be written as

$$q' = a' + ib' + jc' + kd' = q_{ref} \cdot q \cdot q_{ref} =$$
$$= (0 + in_x + jn_y + kn_z) \cdot (0 + ix + jy + kz) \cdot (0 + in_x + jn_y + kn_z) =$$
$$= ((\langle \mathbb{i}, \mathbb{n} \rangle) \cdot ((\langle \mathbb{i}, \mathbb{x} \rangle) \cdot ((\langle \mathbb{i}, \mathbb{n} \rangle) .$$

• • •

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

- **Laplacian eigenfunctions and neural networks:...** The compact notations $\langle \mathbf{i}, \mathbf{x} \rangle$ and $\langle \mathbf{i}, \mathbf{n} \rangle$ define the "inner products" $\langle (i,j,k), (n_x, n_y, n_z) \rangle = i n_x + j n_y + k n_z$ and $\langle (i,j,k), (x,y,z) \rangle = ix + jy + kz$. We now compute q':

$$q' = (i n_x + j n_y + k n_z) \cdot (ix + jy + kz) \cdot (i n_x + j n_y + k n_z) =$$

$$= (i n_x + j n_y + k n_z) \cdot (- x n_x + k x n_y - j x n_z - k y n_x - y n_y + i y n_z + j z n_x - i z n_y - z n_z) =$$

$$\begin{aligned}
= & - i x n_x^2 - j x n_x n_y - k x n_x n_z \\
& + j y n_x^2 - i y n_x n_y \underline{- y n_x n_z} \\
& + k z n_x^2 \underline{+ z n_x n_y} - i z n_x n_z \\
& - j x n_y n_x + i x n_y^2 \underline{+ x n_y n_z} \\
& - i y n_y n_x - j y n_y^2 - k y n_y n_z \\
& \underline{- z n_y n_x} + k z n_y^2 - j z n_y n_z \\
& - k x n_z n_x \underline{- x n_z n_y} + i x n_z^2 \\
& \underline{+ y n_z n_x} - k y n_z n_y + j y n_z^2 \\
& - i z n_z n_x - j z n_z n_y - k z n_z^2 =
\end{aligned}$$

$$\begin{aligned}
= & \overbrace{- y n_x n_z + z n_x n_y + x n_y n_z - z n_y n_x - x n_z n_y + y n_z n_x}^{=0} \\
& + i (- x n_x^2 - y n_x n_y - z n_x n_z + x n_y^2 - y n_y n_x + x n_z^2 - z n_z n_x) \\
& + j (- x n_x n_y + y n_x^2 - x n_y n_x - y n_y^2 - z n_y n_z + y n_z^2 - z n_z n_y) \\
& + k (- x n_x n_z + z n_x^2 - y n_y n_z + z n_y^2 - x n_z n_x - y n_z n_y - z n_z^2) =
\end{aligned}$$

$$\begin{aligned}
= & \, i (x (- n_x^2 + n_y^2 + n_z^2) - 2 y n_x n_y - 2 z n_x n_z) \\
& + j (y (n_x^2 - n_y^2 + n_z^2) - 2 x n_y n_x - 2 z n_y n_z) \\
& + k (z (n_x^2 + n_y^2 - n_z^2) - 2 x n_z n_x - 2 y n_z n_y) = \ldots
\end{aligned}$$

$$\left(\underline{n_x^2 + n_y^2 + n_z^2 = 1} \Rightarrow \begin{aligned} - n_x^2 + n_y^2 + n_z^2 &= 1 - 2 n_x^2, \\ n_x^2 - n_y^2 + n_z^2 &= 1 - 2 n_y^2, \\ n_x^2 + n_y^2 - n_z^2 &= 1 - 2 n_z^2 \end{aligned} \right) \quad = \ldots$$

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:...

$$= i(x(1-2n_x^2)-2(yn_xn_y+zn_xn_z)) + j(y(1-2n_y^2)-2(xn_yn_x+zn_yn_z)) + k(z(1-2n_z^2)-2(xn_zn_x+yn_zn_y)) =$$

$$= i(x - 2xn_x^2 - 2(yn_xn_y+zn_xn_z)) + j(y - 2yn_y^2 - 2(xn_yn_x+zn_yn_z)) + k(z - 2zn_z^2 - 2(xn_zn_x+yn_zn_y)) =$$

$$= (ix+jy+kz) - 2(ixn_x^2+jyn_y^2+kzn_z^2) - 2(i(yn_xn_y+zn_xn_z)+j(xn_yn_x+zn_yn_z) + k(xn_zn_x+yn_zn_y)) =$$

$$= q - v - w \mathrel{\widehat{=}} \mathbb{p} - \vec{\mathbb{v}} - \vec{\mathbb{w}} \; .$$

In order to show that $q' = q - v - w$ is the correct reflection result in quaternion representation – and to understand the meaning of $-v$ and $-w$ – one must consider the underlying geometry and compute $\mathbb{p}'$ by reflecting $\mathbb{p}$ with respect to the implicitly defined reflection plane. This plane is shown in the figure on page 21 (1-14-2024); its implicit definition is $\tilde{x}n_x + \tilde{y}n_y + \tilde{z}n_z = 0$, where $\tilde{\mathbb{x}} = (\tilde{x}, \tilde{y}, \tilde{z})^T$ is a point in the plane (viewed as a positional vector) and $\mathbb{n} = (n_x, n_y, n_z)^T$ is the plane's unit "outward" normal vector, i.e., $\langle \tilde{\mathbb{x}}, \mathbb{n} \rangle = 0$. The parametrically defined line $\mathbb{x}(t) = \mathbb{p} + t\mathbb{n}$ can be used to compute the perpendicular projection of $\mathbb{p}$ into the reflection plane, and subsequently compute $\mathbb{p}'$, by intersecting $\langle \tilde{\mathbb{x}}, \mathbb{n} \rangle = 0$ and $\mathbb{x}(t)$.

...

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... We obtain the intersection point by inserting the parametric expressions $x(t)$, $y(t)$ and $z(t)$ from the line definition $\mathbb{x}(t) = (x(t), y(t), z(t))^T$ into the implicit definition of the plane:

$$(x + t n_x) n_x + (y + t n_y) n_y + (z + t n_z) n_z = 0$$

$$\Rightarrow \quad t \underbrace{(n_x^2 + n_y^2 + n_z^2)}_{=1} = -(x n_x + y n_y + z n_z)$$

$$\Rightarrow \quad t = -(x n_x + y n_y + z n_z).$$

Thus, the perpendicular projection of $\mathbb{p}$ into the reflection plane is the point $\hat{\mathbb{p}} = \mathbb{p} - \langle \mathbb{p}, \mathbb{n} \rangle \mathbb{n}$, since the t-value for point $\mathbb{p} = (x, y, z)^T$ is given by the value of $t = -\langle \mathbb{p}, \mathbb{n} \rangle$. Thus, the point $\mathbb{p}'$ "on the other side of the reflection plane is

$$\begin{aligned}\mathbb{p}' &= \mathbb{p} - 2\langle \mathbb{p}, \mathbb{n} \rangle \mathbb{n} = \mathbb{p} - 2(x n_x + y n_y + z n_z)\,\mathbb{n} = \\ &= (x, y, z)^T - 2(x n_x + y n_y + z n_z)(n_x, n_y, n_z)^T = \\ &= \big(x - 2x n_x^2 - 2 n_x (y n_y + z n_z), \\ &\qquad y - 2y n_y^2 - 2 n_y (x n_x + z n_z), \\ &\qquad z - 2z n_z^2 - 2 n_z (x n_x + y n_y)\big)^T = \\ &= (x, y, z)^T - 2(x n_x^2 + y n_y^2 + z n_z^2)^T \\ &\qquad - 2\big(n_x (y n_y + z n_z) + n_y (x n_x + z n_z) \\ &\qquad + n_z (x n_x + y n_y)\big)^T \\ &= \mathbb{p} - \vec{\mathbb{v}} - \vec{\mathbb{w}} \mathrel{\hat{=}} q - v - w.\end{aligned}$$

These calculations show that the product $q_{ref} \cdot q \cdot q_{ref}$ is a correct representation of the desired reflection and that $q_{ref} \cdot q \cdot q_{ref} = q - v - w = q' \mathrel{\hat{=}} \mathbb{p}' = \mathbb{p} - \vec{\mathbb{v}} - \vec{\mathbb{w}}$.

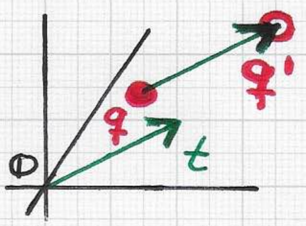
...

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

- **Laplacian eigenfunctions and neural networks:...** The following list of quaternion expressions summarizes the fundamental geometric transformations in 3D space:

- **Translation:**

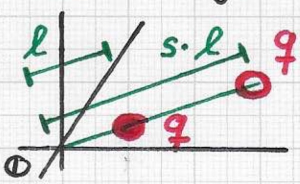
$$q = 0 + ix + jy + kz$$



$$\mapsto q' = q + t = q + (0 + it_x + jt_y + kt_z)$$
$$= (ix + it_x + jy + jt_y + kz + kt_z)$$
$$= 0 + i(x + t_x) + j(y + t_y) + k(z + t_z)$$
$$= 0 + ix' + jy' + kz'$$

- **Scaling:**

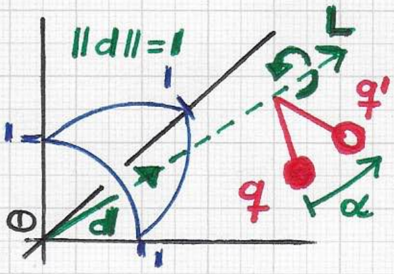
$$q = 0 + ix + jy + kz$$



$$\mapsto q' = s \cdot q = (s + i0 + j0 + k0) \cdot q$$
$$= s \cdot (0 + ix + jy + kz) = isx + jsy + ksz$$
$$= 0 + ix' + jy' + kz'$$

- **Rotation:**

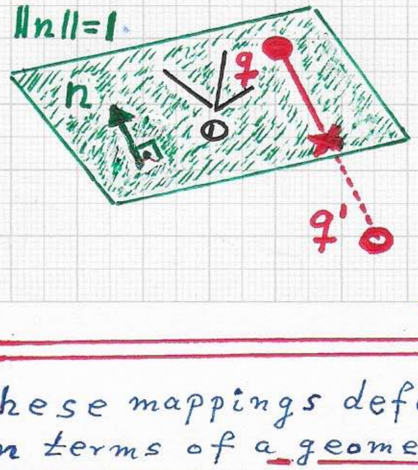
$$q = 0 + ix + jy + kz, \quad d = id_x + jd_y + kd_z$$



$$\mapsto q' = q_{rot} \cdot q \cdot \overline{q_{rot}}$$
$$= \left(c\tfrac{\alpha}{2} + (id_x + jd_y + kd_z)\, s\tfrac{\alpha}{2}\right) \cdot q \cdot \overline{q_{rot}}$$
$$= \left(c\tfrac{\alpha}{2} + \langle \mathbf{i}, \mathbf{d}\rangle s\tfrac{\alpha}{2}\right)\left(\langle \mathbf{i}, \mathbf{x}\rangle\right)\left(c\tfrac{\alpha}{2} - \langle \mathbf{i}, \mathbf{d}\rangle s\tfrac{\alpha}{2}\right)$$
$$= 0 + ix' + jy' + kz'$$

- **Reflection:**

$$q = 0 + ix + jy + kz, \quad n = in_x + jn_y + kn_z$$



$$\mapsto q' = q_{ref} \cdot q \cdot q_{ref}$$
$$= (0 + in_x + jn_y + kn_z) \cdot q \cdot q_{ref}$$
$$= (\langle \mathbf{i}, \mathbf{n}\rangle) \cdot (\langle \mathbf{i}, \mathbf{x}\rangle) \cdot (\langle \mathbf{i}, \mathbf{n}\rangle)$$
$$= 0 + ix' + jy' + kz'$$

Note. $\langle \mathbf{i}, \mathbf{u}\rangle = iu_x + ju_y + ku_z$; $\mathbf{u} = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}$.

These mappings define "basic transformations" in 3D space in terms of a geometric algebra. ...