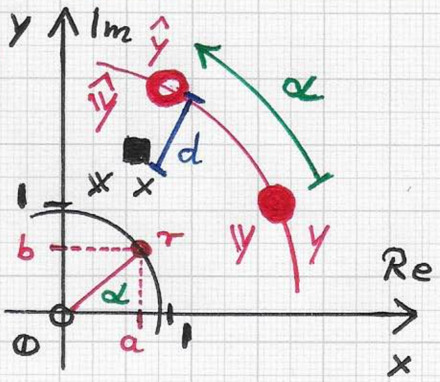
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• **Laplacian eigenfunctions and neural networks:...** First, we describe how to obtain the optimal rotation angle α necessary for applying a rotation to a given point — such that the resulting image point has minimal distance to another given point. The left figure illustrates this scenario.



Point $\mathbb{y} = (y_1, y_2)^T \hat{=}\, y = y_1 + i y_2$ must be rotated (around the origin $\mathbb{0}$) such that the resulting image point $\hat{\mathbb{y}} = (\hat{y}_1, \hat{y}_2)^T \hat{=}\, \hat{y} = \hat{y}_1 + i\hat{y}_2$ has minimal distance d from the point $\mathbb{x} = (x_1, x_2)^T \hat{=}\, x = x_1 + i x_2$. Thus, the objective is to calculate the optimal rotation angle α, i.e., the optimal rotor $r = a + ib$, $a^2 + b^2 = 1$, that defines the rotation mapping $\mathbb{y}$ to $\hat{\mathbb{y}}$ (y to $\hat{y}$). By employing the standard geometrical approach using matrix notation, one obtains:

$$\hat{\mathbb{y}} = \begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \end{bmatrix} = \begin{bmatrix} c\alpha & -s\alpha \\ s\alpha & c\alpha \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1 c\alpha - y_2 s\alpha \\ y_1 s\alpha + y_2 c\alpha \end{bmatrix}.$$

Thus, the resulting squared distance d^2 between $\mathbb{x}$ and image point $\hat{\mathbb{y}}$ is

$$\begin{aligned} d^2 &= (x_1 - (y_1 c\alpha - y_2 s\alpha))^2 + (x_2 - (y_1 s\alpha + y_2 c\alpha))^2 = \\ &= x_1^2 - 2(y_1 c\alpha - y_2 s\alpha)x_1 + (y_1 c\alpha - y_2 s\alpha)^2 + x_2^2 - 2(y_1 s\alpha + y_2 c\alpha)x_2 + (y_1 s\alpha + y_2 c\alpha)^2 = \\ &= x_1^2 - 2y_1 c\alpha x_1 + 2y_2 s\alpha x_1 + y_1^2 c^2\alpha - 2y_1 y_2 c\alpha s\alpha + y_2^2 s^2\alpha \\ &\quad + x_2^2 - 2y_1 s\alpha x_2 - 2y_2 c\alpha x_2 + y_1^2 s^2\alpha + 2y_1 y_2 s\alpha c\alpha + y_2^2 c^2\alpha = \\ &= \ldots \end{aligned}$$

...

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

- **Laplacian eigenfunctions and neural networks:...**

$$= x_1^2 + x_2^2 + y_1^2 + y_2^2 - 2c\alpha(x_1 y_1 + x_2 y_2) + 2s\alpha(x_1 y_2 - x_2 y_1).$$

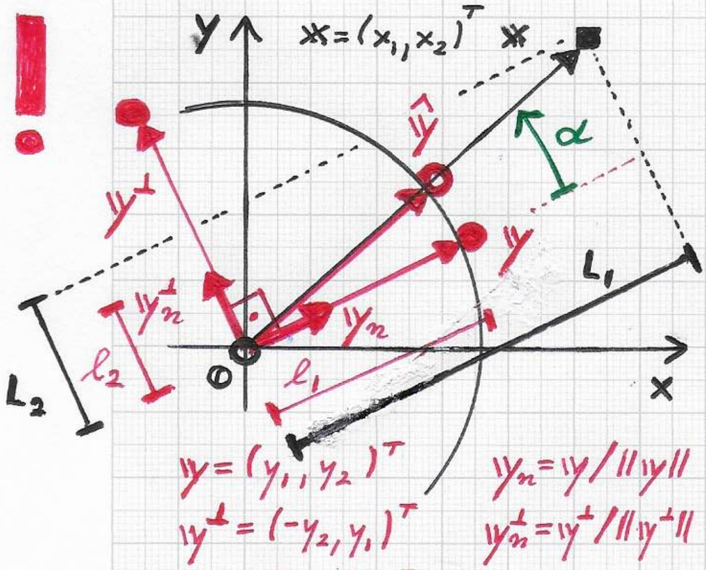
Thus, by defining $D = D(\alpha) = d^2$ the condition for D to be extremal is $\frac{d}{d\alpha} D(\alpha) = 2s\alpha(x_1 y_1 + x_2 y_2) + 2c\alpha(x_1 y_2 - x_2 y_1) = 0$. This condition is equivalent to

$$s\alpha(x_1 y_1 + x_2 y_2) = c\alpha(x_2 y_1 - x_1 y_2)$$

$$\Leftrightarrow \frac{s\alpha}{c\alpha} = \tan\alpha = \frac{x_2 y_1 - x_1 y_2}{x_1 y_1 + x_2 y_2} \Leftrightarrow \cot\alpha = \frac{x_1 y_1 + x_2 y_2}{x_2 y_1 - x_1 y_2}$$

$$\Rightarrow \alpha = \arctan \frac{x_2 y_1 - x_1 y_2}{x_1 y_1 + x_2 y_2} \text{ or } \alpha = \operatorname{arccot} \frac{x_1 y_1 + x_2 y_2}{x_2 y_1 - x_1 y_2}.$$

- **Note.** Since the α-value can be computed by the arctan and arccot functions, it is possible to avoid the division-by-zero case. Also, one must keep in mind the second derivative of $D(\alpha)$, given by $\frac{d^2}{d\alpha^2} D(\alpha) = 2c\alpha(x_1 y_1 + x_2 y_2) + 2s\alpha(x_2 y_1 - x_1 y_2)$.



$\mathbf{y} = (y_1, y_2)^T$ $\quad \mathbf{y}_n = \mathbf{y}/\|\mathbf{y}\|$

$\mathbf{y}^\perp = (-y_2, y_1)^T$ $\quad \mathbf{y}_n^\perp = \mathbf{y}^\perp/\|\mathbf{y}^\perp\|$

$\hat{\mathbf{y}} = (\hat{y}_1, \hat{y}_2)^T$ $\quad L_1 = \langle \mathbf{x}, \mathbf{y} \rangle / \|\mathbf{y}\|$

$L_2 = \langle \mathbf{x}, \mathbf{y}^\perp \rangle / \|\mathbf{y}\|$

The left figure allows one to understand the simple geometry defining this 2D scenario. The following equations hold for the shown variables:

$$\hat{\mathbf{y}} = \ell_1 \mathbf{y}_n + \ell_2 \mathbf{y}_n^\perp = \frac{1}{\|\mathbf{y}\|}(\ell_1 \mathbf{y} + \ell_2 \mathbf{y}^\perp).$$

$$\cos\alpha = \langle \mathbf{x}, \mathbf{y} \rangle / (\|\mathbf{x}\| \|\mathbf{y}\|) = \ell_1 / \|\mathbf{y}\|$$

$$\Rightarrow \ell_1 = \langle \mathbf{x}, \mathbf{y} \rangle / \|\mathbf{x}\|.$$

$$\ell_2 / \ell_1 = L_2 / L_1 \Rightarrow \ell_2 = \ell_1 \cdot L_2 / L_1$$

$$\Rightarrow \ell_2 = \langle \mathbf{x}, \mathbf{y}^\perp \rangle / \|\mathbf{x}\|.$$

...

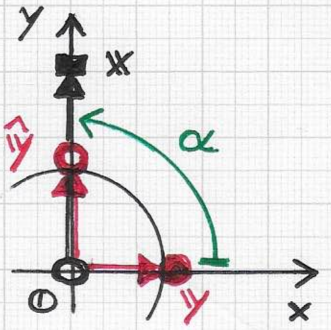
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks: ... $\Rightarrow \hat{\mathbb{y}} = \frac{1}{\|\mathbb{x}\|\,\|\mathbb{y}\|}\left(\langle \mathbb{x},\mathbb{y}\rangle \mathbb{y} + \langle \mathbb{x},\mathbb{y}^{\perp}\rangle \mathbb{y}^{\perp}\right) =$

$$= \frac{1}{\|\mathbb{x}\|\,\|\mathbb{y}\|}\left((x_1y_1 + x_2y_2)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + (-x_1y_2 + x_2y_1)\begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}\right) =$$

$$= \frac{1}{\|\mathbb{x}\|\,\|\mathbb{y}\|}\begin{pmatrix} x_1\,(y_1^2+y_2^2) \\ x_2\,(y_1^2+y_2^2) \end{pmatrix} = \frac{y_1^2+y_2^2}{\|\mathbb{x}\|\,\|\mathbb{y}\|}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{\|\mathbb{y}\|}{\|\mathbb{x}\|}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

This is the interpretation of this result: The vector $\hat{\mathbb{y}}$ is the vector $\mathbb{x}$ scaled by the factor $\|\mathbb{y}\| / \|\mathbb{x}\|$, i.e., the ratio of the lengths of vectors $\mathbb{y}$ and $\mathbb{x}$.

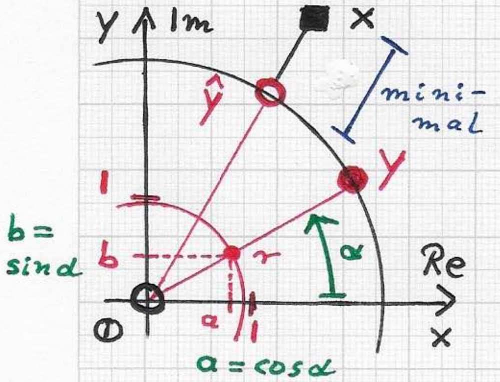


The left figure shows a simple example. The points/vectors $\mathbb{y} = (1,0)^T$ and $\mathbb{x} = (0,2)^T$ are given. The optimal rotation angle α is defined by the arctan and/or the arccot functions. Here, we obtain the two values $\alpha = \arctan\left((x_2y_1 - x_1y_2) / (x_1y_1 + x_2y_2)\right) = \arctan(2/0)$, which cannot be used for this singular case; the alternate formula yields the value $\alpha = \operatorname{arccot}\left((x_1y_1 + x_2y_2)/(x_2y_1 - x_1y_2)\right) = \operatorname{arccot}(0/2) = \operatorname{arccot}(0) = \pi/2$, for $\alpha \in [0, 2\pi]$. Further the second derivative value is $\frac{d^2}{d\alpha^2} D(\alpha) = 2\,c\alpha\,(x_1y_1 + x_2y_2) + 2\,s\alpha\,(x_2y_1 - x_1y_2) = 2\,c\frac{\pi}{2}\,(0\cdot 1 + 2\cdot 0) + 2\,s\frac{\pi}{2}\,(2\cdot 1 - 0\cdot 0) = 0\cdot 0 + 2\cdot 2 = 4 > 0$. Thus, the rotation angle $\alpha = \pi/2$ yields the point $\hat{\mathbb{y}}$ with minimal distance to $\mathbb{x}$, and $\hat{\mathbb{y}} = (0,1)^T$.

...

OBJECT AND MATERIAL EIGENFUNCTIONS - Cont'd.

• **Laplacian eigenfunctions and neural networks:...** Next, we briefly summarize the main steps for solving the optimization problem sketched in the figure on page 21 (2-7-2024) using complex numbers. The essential variables are also shown in the left figure. The variables are:



$x = x_1 + i x_2$, $y = y_1 + i y_2$, $\hat{y} = \hat{y}_1 + i \hat{y}_2$

The rotor $r = a + ib$, $a^2 + b^2 = 1$, must be determined such that $\hat{y}$ has minimal distance to x. The following equations and sequence of steps lead to the result:

$$\hat{y} = r \cdot y = (a+ib)(y_1 + i y_2) = (a y_1 - b y_2) + i (a y_2 + b y_1).$$

$$\Rightarrow x - \hat{y} = (x_1 - (a y_1 - b y_2)) + i (x_2 - (a y_2 + b y_1)).$$

$$\Rightarrow \|x - \hat{y}\|^2 = (x_1 - (a y_1 - b y_2))^2 + (x_2 - (a y_2 + b y_1))^2 = \ldots$$

$$= x_1^2 - 2 a x_1 y_1 + 2 b x_1 y_2 + a^2 y_1^2 - 2ab\, y_1 y_2 + b^2 y_2^2$$

$$+ x_2^2 - 2 a x_2 y_2 - 2 b x_2 y_1 + a^2 y_2^2 + 2ab\, y_2 y_1 + b^2 y_1^2 =$$

$$= x_1^2 + x_2^2 + a^2 (y_1^2 + y_2^2) + b^2 (y_1^2 + y_2^2)$$

$$- 2a (x_1 y_1 + x_2 y_2) + 2b (x_1 y_2 - x_2 y_1) =$$

$$\overset{a^2+b^2=1}{=} x_1^2 + x_2^2 + y_1^2 + y_2^2 - 2a (x_1 y_1 + x_2 y_2) + 2b (x_1 y_2 - x_2 y_1) =$$

$$= x_1^2 + x_2^2 + y_1^2 + y_2^2 - 2 (x_1 y_1 + x_2 y_2) \cos\alpha + 2 (x_1 y_2 - x_2 y_1) \sin\alpha$$

$$= D(\alpha). \quad (D = \text{squared distance})$$

$\Rightarrow$ Necessary condition for an extremum:

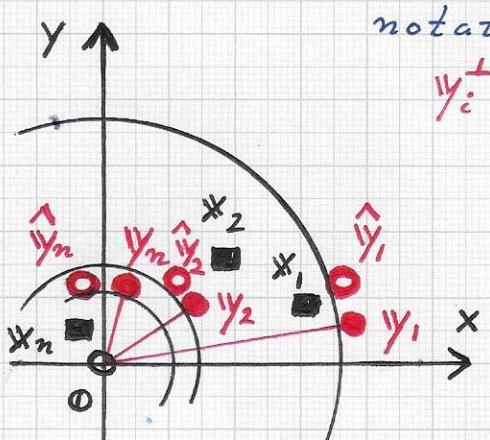
$$\frac{d}{d\alpha} D(\alpha) = +2 (x_1 y_1 + x_2 y_2) \sin\alpha + 2 (x_1 y_2 - x_2 y_1) \cos\alpha = 0$$

$$\Leftrightarrow \quad s\alpha (x_1 y_1 + x_2 y_2) = c\alpha (x_1 y_2 - x_2 y_1).$$

$\Rightarrow$ This is the same condition derived on page 22 (2-8-2024) for the "standard" matrix multiplication. ...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• **Laplacian eigenfunctions and neural networks:...** The more general problem is concerned with mapping a set of points in an optimal way. Given a set $\{\mathbf{y}_i\}_{i=1}^n$ and a set $\{\mathbf{x}_i\}_{i=1}^n$ (point sets), one wants to determine the best-possible rotation of the set $\{\mathbf{y}_i\}_{i=1}^n$, generating the rotated set $\{\hat{\mathbf{y}}_i\}_{i=1}^n$, such the sum of the squared distances $\sum_{i=1}^n \|\mathbf{x}_i - \hat{\mathbf{y}}_i\|^2$ is minimal. It is assumed that the given two sets are both "clustered around the origin" and have (nearly) "the same scale"; thus, one is essentially interested in a rotation that transforms $\{\mathbf{y}_i\}_{i=1}^n$ such that $\{\hat{\mathbf{y}}_i\}_{i=1}^n$ has (nearly) "the same orientation" as $\{\mathbf{x}_i\}_{i=1}^n$. For this general setting, we use the notation $\mathbf{x}_i = (x_1^i, x_2^i)^T$, $\mathbf{y}_i = (y_1^i, y_2^i)^T$, $\mathbf{y}_i^{\perp} = (y_1^{\perp,i}, y_2^{\perp,i})^T = (-y_2^i, y_1^i)^T$ and $\hat{\mathbf{y}}_i = (\hat{y}_1, \hat{y}_2)^T$. Using the definitions provided on pages 21–22 (2–7/8–2024), the squared distance for the point pair $\mathbf{x}_i$ and $\hat{\mathbf{y}}_i$ is $d_i^2 = \|\mathbf{x}_i\|^2 + \|\mathbf{y}_i\|^2 - 2c\alpha\langle \mathbf{x}_i, \mathbf{y}_i\rangle - 2s\alpha\langle \mathbf{x}_i, \mathbf{y}_i^{\perp}\rangle$, see left figure.



Consequently, the function to be minimized is

$$D(\alpha) = \sum_{i=1}^n d_i^2 = \sum_{i=1}^n \left(\|\mathbf{x}_i\|^2 + \|\mathbf{y}_i\|^2 - 2c\alpha\langle \mathbf{x}_i, \mathbf{y}_i\rangle - 2s\alpha\langle \mathbf{x}_i, \mathbf{y}_i^{\perp}\rangle\right) =$$
$$= \sum_{i=1}^n \|\mathbf{x}_i\|^2 + \sum_{i=1}^n \|\mathbf{y}_i\|^2 - 2c\alpha\sum_{i=1}^n \langle \mathbf{x}_i, \mathbf{y}_i\rangle - 2s\alpha\sum_{i=1}^n \langle \mathbf{x}_i, \mathbf{y}_i^{\perp}\rangle.$$

Again, a necessary condition for $D(\alpha)$ to be extremal is $d/d\alpha\, D(\alpha) = 0$.

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