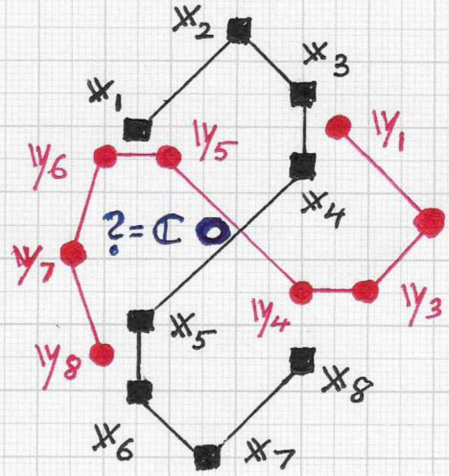
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

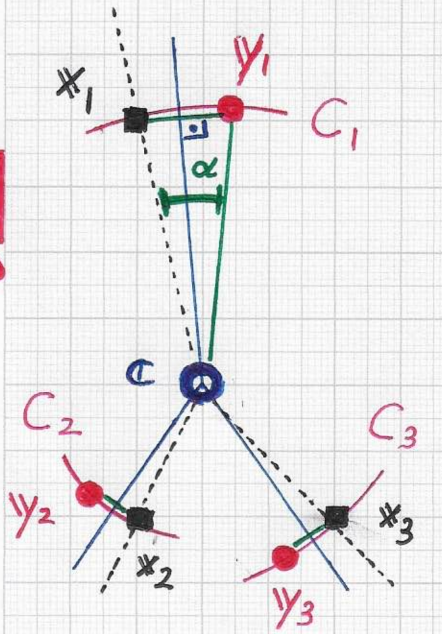
• **Laplacian eigenfunctions and neural networks:...**



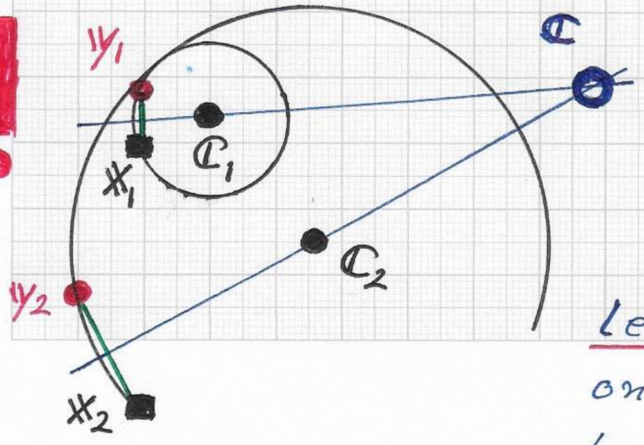
The "two-step method" for determining a center of rotation, $\mathbb{C}$, and subsequently a rotation angle α is now considered in more detail. The left figure shows two point sets, $\{y_i\}_{i=1}^n$ and $\{x_i\}_{i=1}^n$, with the goal of mapping y_i to $\hat{y}_i$ such that the n points in the image set $\{\hat{y}_i\}_{i=1}^n$ are as close as possible to the points in the target point set $\{x_i\}_{i=1}^n$ — in a sum-of-squared-distances sense. The point sets shown in the above figure resemble the "shape of 2"; if one wanted the "shape of S" to be recognized as a reflected version of '2', one would have to employ the approach described on pages 2-4 (2/11-12/2024), treating a point set for a right- and left-handed coordinate system. Further, the above figure shows just **ONE** center of rotation, $\mathbb{C}$, that will be used to determine the optimal rotation angle. The two point sets in the figure have two point set centers/centroids/means that already must be "very close" to each other. Nevertheless, they are generally different, and one must properly define a single **unique** center $\mathbb{C}$. ...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks: ... In the following, we describe a possibility for establishing the needed single unique center of rotation, $\mathbb{C}$.



The left figure illustrates a "perfect example" to explain principles used for the construction of $\mathbb{C}$. Here, the target points are $\mathbb{x}_1 = (0,6)^T$, $\mathbb{x}_2 = (0,0)^T$ and $\mathbb{x}_3 = (3,0)^T$. Their centroid is the shown rotation center $\mathbb{C} = (1,2)^T$. The points to be rotated are the points $\mathbb{y}_1$, $\mathbb{y}_2$ and $\mathbb{y}_3$. In this "perfect example," the three circles C_1, C_2 and C_3 are concentric, sharing $\mathbb{C}$ as their common center. Further, $\mathbb{y}_i$ and $\mathbb{x}_i$ both lie on circle C_i; and a rotation by one unique angle α ($=20°$), around the center $\mathbb{C}$, maps $\mathbb{y}_i$ to $\mathbb{x}_i$ — implying a total error of **zero**.

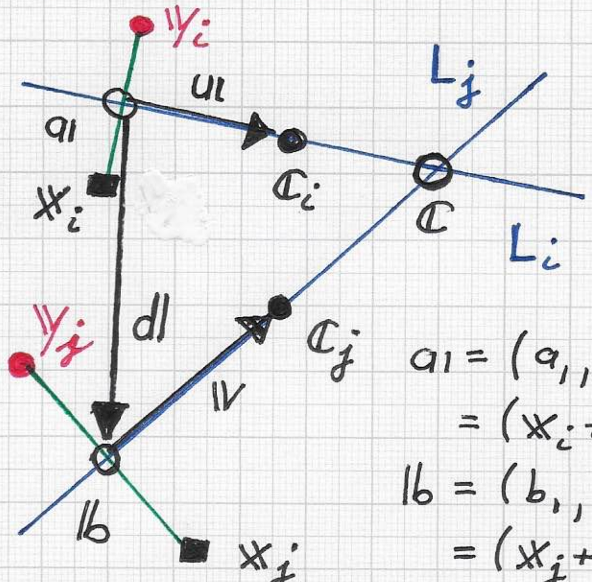


Generally, the point pairs $\mathbb{y}_i$ and $\mathbb{x}_i$ do not lie on concentric circles, and one must construct a best-possible center $\mathbb{C}$. The left figure shows the principle one can use in the case of two point pairs to construct $\mathbb{C}$. ...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:...

• **Note.** We are primarily interested in the 3D case. Nevertheless, we briefly consider the 2D case as it points out some related geometric principles. In the 2D case, one must compute the intersection of all possible pairs of lines L_i and L_j, called $\mathbf{c}$ in the left figure. The midpoint of the segment $\overline{\mathbf{x}_i \mathbf{y}_i}$ is called $\mathbf{a}$, and the midpoint of segment $\overline{\mathbf{x}_j \mathbf{y}_j}$ is called $\mathbf{b}$. Together with the direction vectors $\mathbf{u}$ and $\mathbf{v}$, they define the two lines L_i and L_j as shown and explained in the figure.



$$\mathbf{a} = (a_1, a_2)^T = (\mathbf{x}_i + \mathbf{y}_i)/2$$

$$\mathbf{b} = (b_1, b_2)^T = (\mathbf{x}_j + \mathbf{y}_j)/2$$

$$\mathbf{u} = (u_1, u_2)^T = \mathbf{c}_i - \mathbf{a}$$

$$\mathbf{v} = (v_1, v_2)^T = \mathbf{c}_j - \mathbf{b}$$

$$\mathbf{d} = (d_1, d_2)^T = \mathbf{b} - \mathbf{a}$$

Lines L_1 and L_2 intersect in point $\mathbf{c}$; $\mathbf{c} = (c_1, c_2)^T$.

One can calculate the point $\mathbf{c}$ as follows:

$$\mathbf{c} = \mathbf{a} + \alpha \mathbf{u} = \mathbf{b} + \beta \mathbf{v} \Rightarrow \alpha \mathbf{u} - \beta \mathbf{v} = \mathbf{d}$$

$$\Rightarrow \begin{bmatrix} u_1 & -v_1 \\ u_2 & -v_2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \Rightarrow D = \begin{vmatrix} u_1 & -v_1 \\ u_2 & -v_2 \end{vmatrix} = -u_1 v_2 + u_2 v_1 = \langle \mathbf{u}, \mathbf{v}^\perp \rangle;$$

$$D_1 = \begin{vmatrix} d_1 & -v_1 \\ d_2 & -v_2 \end{vmatrix} = -d_1 v_2 + d_2 v_1 = \langle \mathbf{d}, \mathbf{v}^\perp \rangle; \quad D_2 = \begin{vmatrix} u_1 & d_1 \\ u_2 & d_2 \end{vmatrix} =$$

$$= -d_1 u_2 + d_2 u_1 = \langle \mathbf{d}, \mathbf{u}^\perp \rangle.$$

Here, $\mathbf{u}^\perp = (-u_2, u_1)^T$ and $\mathbf{v}^\perp = (-v_2, v_1)^T$. ...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks: … The determinants $\mathcal{D}$, $\mathcal{D}_1$ and $\mathcal{D}_2$ define the solution of the linear system:

$$\alpha = \mathcal{D}_1 / \mathcal{D} = \langle \mathbb{d}, \mathbb{v}^{\perp} \rangle / \langle \mathbb{u}, \mathbb{v}^{\perp} \rangle \text{ and}$$
$$\beta = \mathcal{D}_2 / \mathcal{D} = \langle \mathbb{d}, \mathbb{u}^{\perp} \rangle / \langle \mathbb{u}, \mathbb{v}^{\perp} \rangle .$$

Thus, the values of parameters α and β are given by ratios of scalar products of $\mathbb{d}$, $\mathbb{u}$, $\mathbb{u}^{\perp}$ and $\mathbb{v}^{\perp}$. One must handle the special case that arises when $\langle \mathbb{u}, \mathbb{v}^{\perp} \rangle = 0$. This case indicates that the vectors $\mathbb{u}$ and $\mathbb{v}$ are multiples of each other and the lines L_i and L_j are parallel, not intersecting in a point. For example, one could simply "ignore" such a parallel line pair for intersection point calculation.

$\cap$	L_1	L_2	L_3	L_4	L_5
L_1		$\mathbb{C}_{1,2}$	$\mathbb{C}_{1,3}$	$\mathbb{C}_{1,4}$	$\mathbb{C}_{1,5}$
L_2			$\mathbb{C}_{2,3}$	$\mathbb{C}_{2,4}$	$\mathbb{C}_{2,5}$
L_3				$\mathbb{C}_{3,4}$	$\mathbb{C}_{3,5}$
L_4					$\mathbb{C}_{4,5}$
L_5					

The number of intersection points for lines $L_1, \dots, L_5$ is (at most) **10**.

The left table shows the (up to) ten intersection points for five lines, using the notation $\mathbb{C}_{i,j}$ for the intersection point of lines L_i and L_j. For n lines, the (maximal) number of intersection points is given as

$$N = \sum_{i=1}^{n-1} i \Longrightarrow N = n(n-1)/2.$$

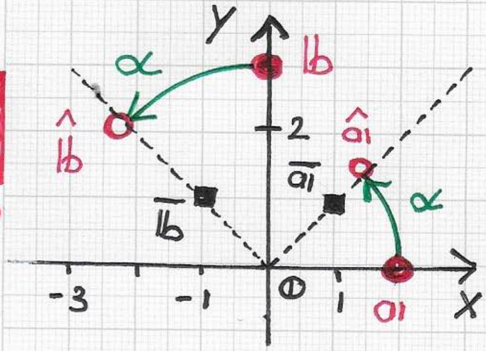
…

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:… Assuming that one can compute $N = n(n-1)/2$ intersection points $\mathbb{C}_{i,j}$ for all possible line pairs L_i and L_j from a set of (non-parallel) lines, i.e., $L_i, L_j \in \{L_k\}_{k=1}^{n}$, $i \neq j$, one can define the needed final average center point as average of the calculated points $\mathbb{C}_{i,j}$:

$$\boxed{\mathbb{C}_{average} = \frac{1}{N} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \mathbb{C}_{i,j} .}$$

The point $\mathbb{C}_{average}$ can now be used as the "best-possible center" when considering all point pairs $\mathbb{x}_i$ and $\mathbb{y}_i$. We describe an example.



$\mathbb{a} = (a_1, a_2)^T$, $\bar{\mathbb{a}} = (\bar{a}_1, \bar{a}_2)^T$
$\mathbb{b} = (b_1, b_2)^T$, $\bar{\mathbb{b}} = (\bar{b}_1, \bar{b}_2)^T$

Example. The left figure shows a simple example where two points, $\mathbb{a}$ and $\mathbb{b}$, must be rotated around the center $\mathbb{C} = \mathbb{0}$ such that the sum of the squared distances between the image points and the given points $\bar{\mathbb{a}}$ and $\bar{\mathbb{b}}$ is minimal. Using the notation $s = \sin\alpha$ and $c = \cos\alpha$, the image points of $\mathbb{a}$ and $\mathbb{b}$ are $(ca_1 - sa_2, sa_1 + ca_2)^T$ and $(cb_1 - sb_2, sb_1 + cb_2)^T$.

• • •