Stratovan

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks: ... With these definitions we can now solve the minimization problem that minimizes the value of $D = d_a^2 + d_b^2 = \|\hat{a} - \bar{a}\|^2 + \|\hat{b} - \bar{b}\|^2$.

These are the calculations in detail:

$$d_a^2 = \left((ca_1 - sa_2) - \bar{a}_1\right)^2 + \left((sa_1 + ca_2) - \bar{a}_2\right)^2 =$$

$$= (ca_1 - sa_2)^2 - 2\bar{a}_1(ca_1 - sa_2) + \bar{a}_1^2$$
$$+ (sa_1 + ca_2)^2 - 2\bar{a}_2(sa_1 + ca_2) + \bar{a}_2^2 =$$

$$= (ca_1)^2 - 2cs\,a_1a_2 + (sa_2)^2 - 2ca_1\bar{a}_1 + 2s\,a_2\bar{a}_1 + \bar{a}_1^2$$
$$+ (sa_1)^2 + 2sc\,a_1a_2 + (ca_2)^2 - 2sa_1\bar{a}_2 - 2c\,a_2\bar{a}_2 + \bar{a}_2^2 =$$

$$= a_1^2\underbrace{(c^2+s^2)}_{1} + a_2^2\underbrace{(s^2+c^2)}_{1} - 2c(a_1\bar{a}_1 + a_2\bar{a}_2)$$
$$+ 2s(a_2\bar{a}_1 - a_1\bar{a}_2) + \bar{a}_1^2 + \bar{a}_2^2 =$$

$$= a_1^2 + a_2^2 + \bar{a}_1^2 + \bar{a}_2^2 - 2c(a_1\bar{a}_1 + a_2\bar{a}_2) + 2s(a_2\bar{a}_1 - a_1\bar{a}_2)$$

$$\Rightarrow d_b^2 = b_1^2 + b_2^2 + \bar{b}_1^2 + \bar{b}_2^2 - 2c(b_1\bar{b}_1 + b_2\bar{b}_2) + 2s(b_2\bar{b}_1 - b_1\bar{b}_2)$$

$$\Rightarrow D = a_1^2 + a_2^2 + \bar{a}_1^2 + \bar{a}_2^2 + b_1^2 + b_2^2 + \bar{b}_1^2 + \bar{b}_2^2$$
$$-2c(a_1\bar{a}_1 + a_2\bar{a}_2 + b_1\bar{b}_1 + b_2\bar{b}_2)$$
$$+2s(a_2\bar{a}_1 - a_1\bar{a}_2 + b_2\bar{b}_1 - b_1\bar{b}_2) = D(\alpha) \longrightarrow MIN$$

$$\Rightarrow \frac{d}{d\alpha}D(\alpha) = 2s(a_1\bar{a}_1 + a_2\bar{a}_2 + b_1\bar{b}_1 + b_2\bar{b}_2)$$
$$+2c(a_2\bar{a}_1 - a_1\bar{a}_2 + b_2\bar{b}_1 - b_1\bar{b}_2) \overset{!}{=} 0$$

$$\Rightarrow \boxed{\frac{s}{c} = \tan\alpha = \frac{a_1\bar{a}_2 - a_2\bar{a}_1 + b_1\bar{b}_2 - b_2\bar{b}_1}{a_1\bar{a}_1 + a_2\bar{a}_2 + b_1\bar{b}_1 + b_2\bar{b}_2}.}$$

(The case $a_1\bar{a}_1 + a_2\bar{a}_2 + b_1\bar{b}_1 + b_2\bar{b}_2 = 0$ is a degenerate, singular case, and special-case treatment is necessary. Specifically, the tan function has poles at $-\pi/2$ and $\pi/2$.)

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■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... It is also possible to express the $\tan\alpha$ value in terms of scalar products of vectors:

$$\boxed{\tan\alpha = \frac{\langle \mathbf{a}, -\bar{\mathbf{a}}^{\perp}\rangle + \langle \mathbf{b}, -\bar{\mathbf{b}}^{\perp}\rangle}{\langle \mathbf{a}, \bar{\mathbf{a}}\rangle + \langle \mathbf{b}, \bar{\mathbf{b}}\rangle} = -\frac{\langle \mathbf{a}, \bar{\mathbf{a}}^{\perp}\rangle + \langle \mathbf{b}, \bar{\mathbf{b}}^{\perp}\rangle}{\langle \mathbf{a}, \bar{\mathbf{a}}\rangle + \langle \mathbf{b}, \bar{\mathbf{b}}\rangle}}.$$

The figure on page 20 (3-24-2024) illustrates an example where the optimal rotation angle is $\alpha = \pi/4$. We can compute this value by using the derived formula for $\tan\alpha$ and the specific points/vectors:

$$\mathbf{a} = (2,0)^T, \quad \bar{\mathbf{a}} = (1,1)^T, \quad \bar{\mathbf{a}}^{\perp} = (-1,1)^T,$$
$$\mathbf{b} = (0,3)^T, \quad \bar{\mathbf{b}} = (-1,1)^T, \quad \bar{\mathbf{b}}^{\perp} = (-1,-1)^T.$$

$\Rightarrow \tan\alpha = -(-2+0-0-3)/(2+0-0+3) = -(-5)/5 = 1.$

$\Rightarrow \alpha = \arctan(1) = \pi/4 \; (= 45°).$

As before, the vectors used are the positional vectors of the corresponding points.

The singular case arises for these values:

$$\mathbf{a} = (2,0)^T, \quad \bar{\mathbf{a}} = (0,1)^T, \quad \bar{\mathbf{a}}^{\perp} = (-1,0)^T,$$
$$\mathbf{b} = (0,3)^T, \quad \bar{\mathbf{b}} = (-2,0)^T, \quad \bar{\mathbf{b}}^{\perp} = (0,-2)^T.$$

$\Rightarrow \tan\alpha = -(-2+0+0-6)/(0+0-0+0) = -(-8)/0.$

$\Rightarrow \tan\alpha = 8/0 = \text{"}+\infty\text{"}. \Rightarrow \alpha = \arctan(\infty) = \pi/2.$

The discussion and the numerical examples uses the origin $\mathbb{O}$ as the center of rotation. This is not restrictive, since the computed optimal center $\mathbb{c}$ can be "turned into the origin by translating all data by $-\mathbb{c}$.

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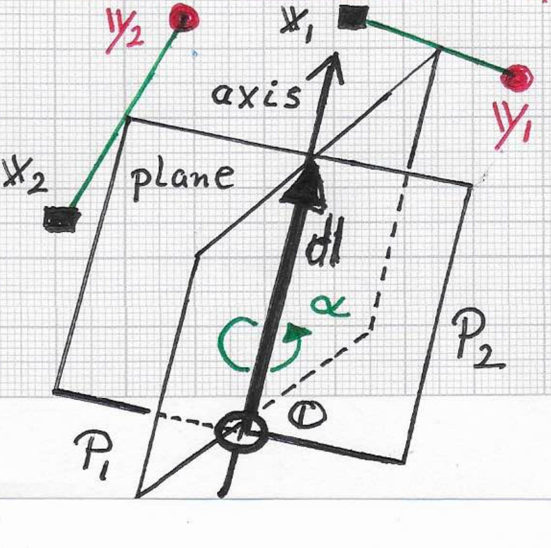
OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

- Laplacian eigenfunctions and neural networks:... In the general setting, where n point pairs $\mathbb{y}_i$ and $\mathbb{x}_i$ must be considered, the function to be minimized is $D = D(\alpha) = d_1^2 + \ldots + d_n^2 = \| \hat{\mathbb{y}}_1 - \mathbb{x}_1 \|^2 + \ldots + \| \hat{\mathbb{y}}_n - \mathbb{x}_n \|^2$. Once the optimal value of α has been computed, one can use, for example, the value of D, or of $(1/n\, D)^{1/2}$ ("RMS error") or $\max\{d_i\}_{i=1}^n$, $d_i = (d_i^2)^{1/2}$, as a basis for the definition and computation of an overall distance measure for the two sets $\{\hat{\mathbb{y}}_i\}_{i=1}^n$ and $\{\mathbb{x}_i\}_{i=1}^n$.
- **Note**. Our main focus is the 3D case. Here the point pairs $\mathbb{y}_i$ and $\mathbb{x}_i$ are points in 3D space, and the goal is to compute an "optimal rotation axis" – defined by a (unit) axis direction vector of a line passing through the origin – for an optimal rotation in quaternion representation.



The left figure is a simplified version of the figure shown on page 7, bottom (2-14-2024). First, one constructs the unit vector $\mathbb{d} = (u, v, w)^T$; second, one computes an optimal value of α.

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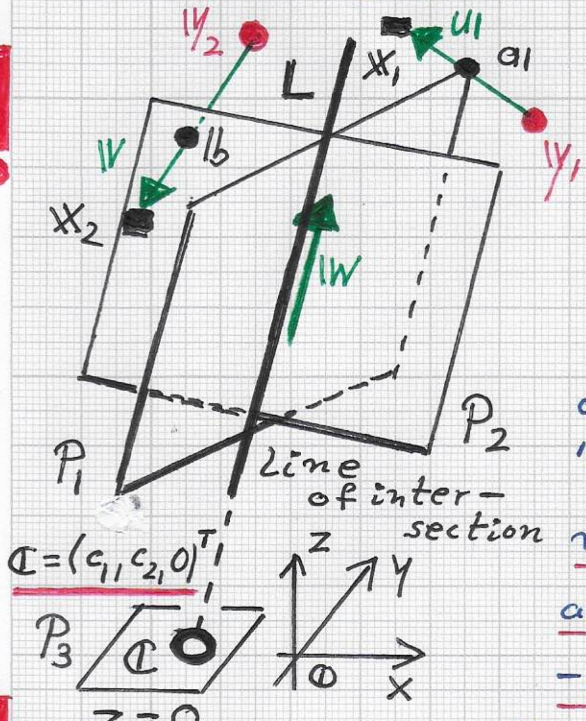
■ OBJECT AND MATERIAL EIGENFUNCTIONS - Cont'd.

• Laplacian eigenfunctions and neural networks:... Again, a rotation in 3D space is written in quaternion notation as $q' = (c\frac{\alpha}{2} + s\frac{\alpha}{2}\langle i, u\rangle)\, q\, (c\frac{\alpha}{2} - s\frac{\alpha}{2}\langle i, u\rangle)$, where a point $(x, y, z)^T$ is represented as $q = 0 + ix + jy + kz = \langle i, x\rangle$, and the inner product $\langle i, u\rangle$ defines $iu + jv + kw$. Further, $c\frac{\alpha}{2} = \cos(\alpha/2)$, $s\frac{\alpha}{2} = \sin(\alpha/2)$, and α is the actual rotation angle. The figure on the previous page only shows a single axis passing through the origin 0 and having direction d, $\|d\| = 1$, and the angle α. In the discussed 2D case, the "best-possible center of rotation" is defined as an average of all possible line-line intersections. The left figure illustrates the 3D case in detail, showing the intersection of the plane pair P_1 and P_2. Plane P_1 passes through a and has the normal $u = x_1 - y_1$; plane P_2 passes through b and has the normal $v = x_2 - y_2$. The cross product $w = u \times v$ defines L's direction...



$z = 0$

$a = (y_1 + x_1)/2 = (a_1, a_2, a_3)^T$

$b = (y_2 + x_2)/2 = (b_1, b_2, b_3)^T$

$u = (u_1, u_2, u_3)^T$

$v = (v_1, v_2, v_3)^T$

$w = u \times v$

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks!... In the 3D case, one considers all possible plane-plane intersection of pairs P_i and P_j, producing intersection lines $L_{i,j}$ and associated direction vectors $\mathbb{w}_{i,j}$; after averaging, the one needed "best-possible rotation axis direction vector $\mathbb{d}$" is obtained. One can employ various plane-plane intersection methods, involving implicit or parametric plane representations. We briefly sketch a method using implicitly defined planes P_1 and P_2 to compute their intersection. One solves the intersection problem as follows:

$P_1: u_1(x-a_1)+u_2(y-a_2)+u_3(z-a_3)=0 \Rightarrow \langle \mathbb{u}, \mathbb{x}\rangle = \langle \mathbb{u}, \mathbb{a}\rangle$

$P_2: v_1(x-b_1)+v_2(y-b_2)+v_3(z-b_3)=0 \Rightarrow \langle \mathbb{v}, \mathbb{x}\rangle = \langle \mathbb{v}, \mathbb{b}\rangle$

P_3: $z=0$ (see figure on previous page)

$\Rightarrow$ Line L intersects plane P_3 $(z=0)$ in $\mathbb{c}=(c_1, c_2, 0)^T$:

$P_1|_{z=0}: u_1 x + u_2 y = \langle \mathbb{u}, \mathbb{a}\rangle$; $P_2|_{z=0}: v_1 x + v_2 y = \langle \mathbb{v}, \mathbb{b}\rangle$

$\Rightarrow \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \langle \mathbb{u}, \mathbb{a}\rangle \\ \langle \mathbb{v}, \mathbb{b}\rangle \end{pmatrix} \Rightarrow c_1 = x = D_1/D,\ c_2 = y = D_2/D$;

$D = u_1 v_2 - u_2 v_1$, $D_1 = v_2\langle \mathbb{u}, \mathbb{a}\rangle - u_2\langle \mathbb{v}, \mathbb{b}\rangle$, $D_2 = u_1\langle \mathbb{v}, \mathbb{b}\rangle - v_1\langle \mathbb{u}, \mathbb{a}\rangle$.

($\Rightarrow L: \mathbb{x}(t) = \mathbb{c} + t\,\mathbb{w}$)

$D=0$ $\Rightarrow$ Use $P_3: x=0$ or $P_3: y=0$

Concerning the quaternion-based rotation notation, only $\mathbb{w}_{i,j}$ is relevant. One can define $\hat{\mathbb{w}}_{i,j} = \mathbb{w}_{i,j}/\|\mathbb{w}_{i,j}\|$ and use all pairs $+\hat{\mathbb{w}}_{i,j}$ and $-\hat{\mathbb{w}}_{i,j}$, for all plane pairs P_i and P_j, to calculate the needed final axis direction vector $\mathbb{d}$, obtained via PCA of $\{+\hat{\mathbb{w}}_{i,j}, -\hat{\mathbb{w}}_{i,j}\}$...