OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... • **Note.** Complex numbers, quaternions, octonions and more general hyper-complex numbers are not "intuitive" when using them for representations and computations inherently related to geometry. Geometric algebra is an area one can understand as an area that closely "links" a driving geometrical problem with a more "intuitive" representation and algebraic operations. The term geometric algebra is often viewed as an area consisting of several closely connected sub-areas — some being very closely related to a geometrical context, some being rather abract and describing high-dimensional geometry. Important sub-areas of geometric algebra (or intimately related) include: Grassmann algebra, quaternions and general hyper-complex algebra (Hamilton et al.), Cayley matrix algebra, Gibbs' vector analysis and vector calculus, Clifford algebra (and its multi-dimensional generalization by Lipschitz). Only some of the aspects and concepts often associated with geometric algebra are of interest in the context of our applications. We focus on the wedge (outer, exterior, progressive) product; bivectors and trivectors; rotors; reflections and rotations. ...

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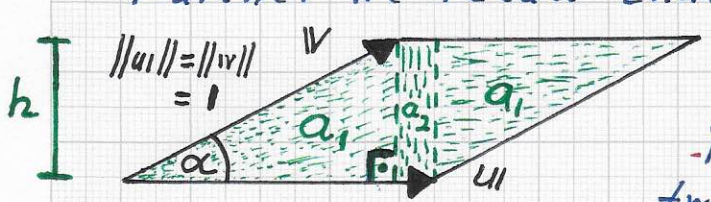
• Laplacian eigenfunctions and neural networks:... Grassman algebra (exterior algebra) uses the so-called exterior product or wedge product, since the wedge symbol '$\wedge$' is used as notation for this product. The wedge product can be used to calculate areas, volumes or hyper-volumes defined by wedge products of 2 vectors determining the area of a parallelogram ($\mathbb{v}_1 \wedge \mathbb{v}_2$), of 3 vectors determining the volume of a parallelepiped ($\mathbb{v}_1 \wedge \mathbb{v}_2 \wedge \mathbb{v}_3$) or of k vectors determining a hyper-volume ($\mathbb{v}_1 \wedge \mathbb{v}_2 \wedge \ldots \wedge \mathbb{v}_k$). These wedge products are called a 2-blade, a 3-blade or a k-blade, respectively. Before describing specific rules and properties of wedge product computations and their geometrical meaning, we discuss the **PARALLELOGRAM**. It serves the purpose of a brief review of closely related calculations. We use the notation $\langle \mathbb{v}_1, \mathbb{v}_2 \rangle$ for the scalar/dot/inner product of two vectors in the following. Further we recall that $\langle \mathbb{v}_1, \mathbb{v}_2 \rangle = \mathbb{v}_1^T \cdot \mathbb{v}_2$.



Area: $A = 2a_1 + a_2$

Height: $h = (1 - \langle \mathbb{u}, \mathbb{v} \rangle^2)^{1/2}$

The left figure shows a parallelogram spanned by two unit vectors, $\mathbb{u}$ and $\mathbb{v}$. Its area is $A = 2\, h \cdot \langle \mathbb{u}, \mathbb{v} \rangle + h\,(1 - \langle \mathbb{u}, \mathbb{v} \rangle) = h$.

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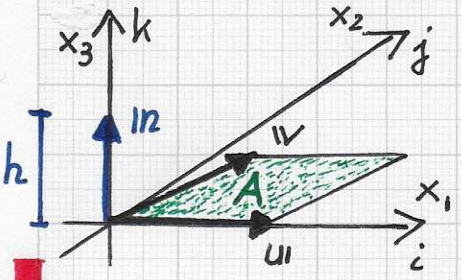
- Laplacian eigenfunctions and neural networks: ...



- We define the two unit vectors as $\mathbb{u} = (u_1, u_2)^T$ and $\mathbb{v} = (v_1, v_2)^T$. The following **2D determinant** also defines the area A:
$A = \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} = u_1 v_2 - u_2 v_1$. The left figure shows the 2D parallelogram scenario embedded in 3D space. It is important to note that the determinant yields a signed value for the area.
- The above figure also shows the vector $\mathbb{n} = (n_1, n_2, n_3)^T$, the normal vector of the plane containing the parallelogram. The sign/orientation of $\mathbb{n}$ and its length h are defined by the "orientation of the parallelogram" and its area, respectively. The normal $\mathbb{n}$ can be formally obtained via the cross product of $\mathbb{u}$ and $\mathbb{v}$ that is defined by the determinant shown to the left.

$$\begin{vmatrix} u_1 & v_1 & i \\ u_2 & v_2 & j \\ 0 & 0 & k \end{vmatrix} = 0i - 0j + (u_1 v_2 - u_2 v_1)k \,\hat{=}\, \begin{bmatrix} 0 \\ 0 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

This "formal" **3D determinant** defines the component coordinates of $\mathbb{n}$ as coefficient of i, j and k. Thus, $\mathbb{n}$ can "point up or down."

*** For the purpose of this discussion, it is relevant to note that the absolute value of the rotation angle, i.e., $|\alpha|$ satisfies $0 \le |\alpha| \le \pi$. ...

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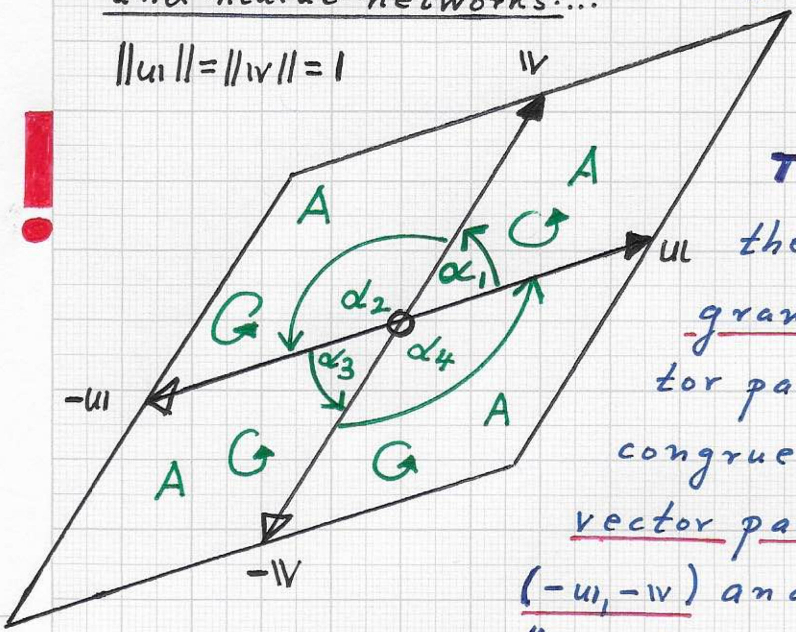
• Laplacian eigenfunctions and neural networks:...

$\|u\| = \|v\| = 1$



• Sign and orientation are important aspects one must keep in mind. The left figure shows the four possible parallelograms implied by the vector pairs spanning these four congruent parallelograms. The vector pairs are (u, v), $(v, -u)$, $(-u, -v)$ and $(-v, u)$. One can think of "the first vector being rotated into the second vector" — by using the positive rotation angles α_1, α_2, $\alpha_3 (=\alpha_1)$ and $\alpha_4 (=\alpha_2)$, respectively. Using the **2D** determinant rule for area computation, the area **A** is defined as

$$A = \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix}, \quad A = \begin{vmatrix} v_1 & -u_1 \\ v_2 & -u_2 \end{vmatrix} = - \begin{vmatrix} v_1 & u_1 \\ v_2 & u_2 \end{vmatrix} = -- \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} = \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix},$$

$$A = \begin{vmatrix} -u_1 & -v_1 \\ -u_2 & -v_2 \end{vmatrix} = -- \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} = \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix},$$

$$A = \begin{vmatrix} -v_1 & u_1 \\ -v_2 & u_2 \end{vmatrix} = - \begin{vmatrix} v_1 & u_1 \\ v_2 & u_2 \end{vmatrix} = -- \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} = \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix}.$$

If one thought of "rotating the second vector into the first vector" — by using clock-wise oriented rotation angles, i.e., $-\alpha_1$, $-\alpha_2$, $-\alpha_3$ and $-\alpha_4$, then the area's sign would be the opposite. For example,

$$\begin{vmatrix} v_1 & u_1 \\ v_2 & u_2 \end{vmatrix} = u_2 v_1 - u_1 v_2 = -(u_1 v_2 - u_2 v_1) = -A.$$

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■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

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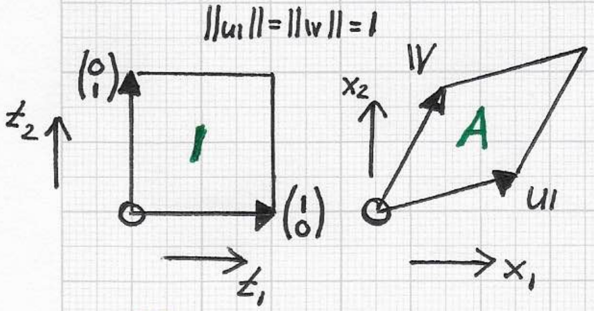
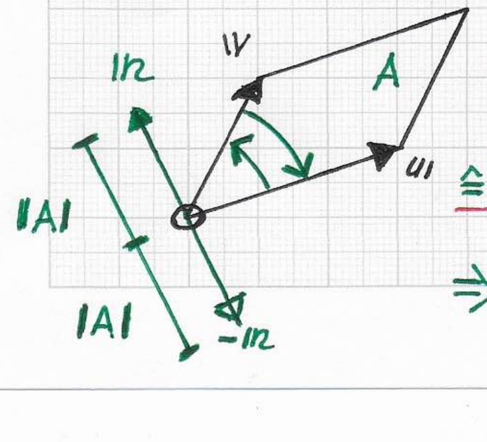


• Another related concept is a "linear deformation of space." The left figure illustrates a linear map that maps $(t_1, t_2)^T$ to $(x_1(t_1,t_2), x_2(t_1,t_2))^T$, without a translation of the origin.

Using vector-valued notation, the point/vector $\mathbf{t} = (t_1, t_2)^T$ is mapped to $\mathbf{x} = \mathbf{x}(\mathbf{t}) = (x_1(t_1,t_2), x_2(t_1,t_2))^T$. Specifically the simple linear deformation shown in the figure is defined by $(1,0)^T \mapsto (u_1, u_2)^T$ and $(0,1)^T \mapsto (v_1, v_2)^T$. In matrix notation, the mapping is written as $\mathbf{x} = M\,\mathbf{t}$, where $M = \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix}$. The basis vectors in the $\mathbf{t}$-domain span a unit square, and their image vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{x}$-space span a parallelogram with area A. Using the laws of vector calculus, the area A of the parallelogram is the value of the Jacobian determinant of the mapping:

$$J = \begin{vmatrix} \partial/\partial t_1\, x_1(t_1,t_2) & \partial/\partial t_2\, x_1(t_1,t_2) \\ \partial/\partial t_1\, x_2(t_1,t_2) & \partial/\partial t_2\, x_2(t_1,t_2) \end{vmatrix} = \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} = u_1 v_2 - u_2 v_1 = \underline{A}.$$



• The cross product demonstrates the relationships between direction, sign, orientation, area, normal and the right-hand rule.

$$\begin{vmatrix} u_1 & v_1 & i \\ u_2 & v_2 & j \\ u_3 & v_3 & k \end{vmatrix} = \begin{matrix} (u_2 v_3 - u_3 v_2)\, i \\ -(u_1 v_3 - u_3 v_1)\, j \\ +(u_1 v_2 - u_2 v_1)\, k \end{matrix}$$

$$\triangleq (u_2 v_3 - u_3 v_2,\ u_3 v_1 - u_1 v_3,\ u_1 v_2 - u_2 v_1)^T = \mathbf{n}$$

$$\Rightarrow \mathbf{n} = \mathbf{u} \times \mathbf{v}, \quad -\mathbf{n} = \mathbf{v} \times \mathbf{u}$$

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