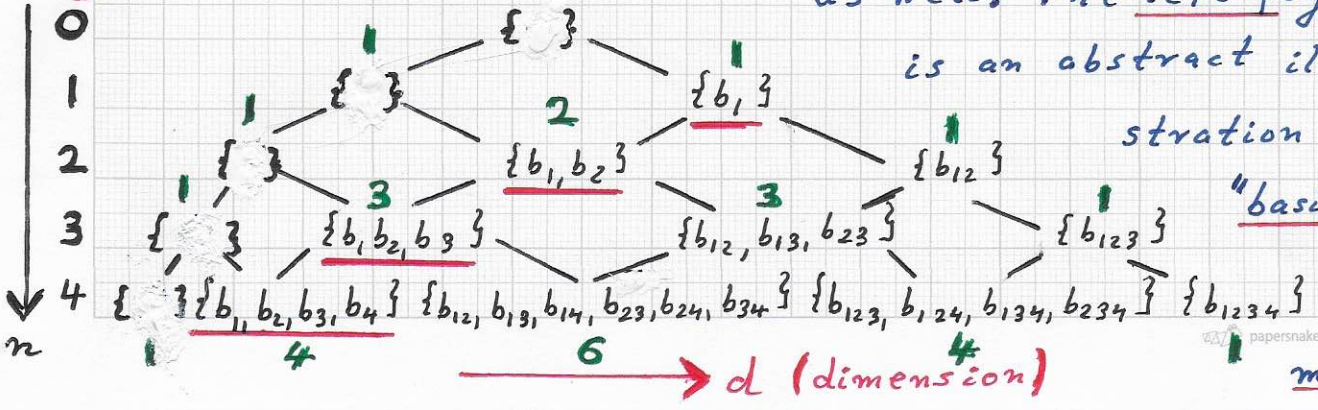
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... Considering the figure at the bottom of the previous page, the "positive normal" $\mathbb{n}$ is obtained by "rotating $\mathbb{u}$ into $\mathbb{v}$" using a positive rotation angle while the "negative normal" $-\mathbb{n}$ is obtained by "rotating $\mathbb{v}$ into $\mathbb{u}$" using a negative rotation angle. The RIGHT-HAND RULE defines the orientation of the "positive normal" that is perpendicular to the parallelogram spanned by $\mathbb{u}$ and $\mathbb{v}$. Orientation also defines the sign of area A. Thus, the signed area has one of the two values:

$$A = \begin{vmatrix} u_1 & v_1 & 1 \\ u_2 & v_2 & 1 \\ u_3 & v_3 & 1 \end{vmatrix} = \begin{matrix} u_2v_3 - u_3v_2 \\ -u_1v_3 + u_3v_1 \\ +u_1v_2 - u_2v_1 \end{matrix} \quad \text{or} \quad A = \begin{vmatrix} v_1 & u_1 & 1 \\ v_2 & u_2 & 1 \\ v_3 & u_3 & 1 \end{vmatrix} = \begin{matrix} u_3v_2 - u_2v_3 \\ -u_3v_1 + u_1v_3 \\ +u_2v_1 - u_1v_2 \end{matrix} .$$

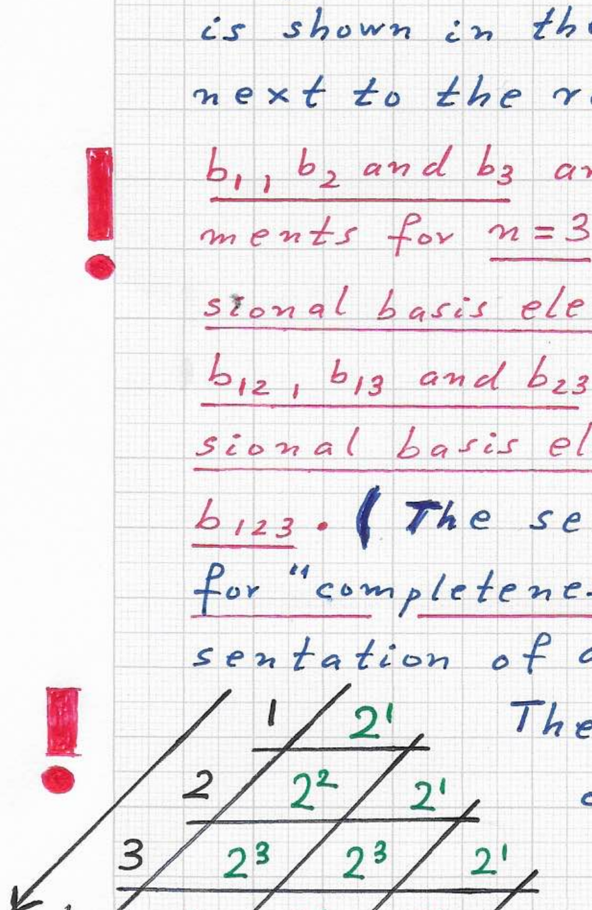
The two possible rotations are also called counter-clockwise rotation (positive) and clockwise rotation (negative). • Dimension, combinations and permutations are crucial as well. The left figure is an abstract illustration of "basis elements" (n) and "dimension"...

$$\binom{n}{d} = \frac{n!}{d!\,(n-d)!}$$



■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... The figure on the previous page shows "sets consisting of d-dimensional basis elements that can be constructed from n one-dimensional basis elements called $b_1, b_2, \ldots, b_n$." The illustration points out the relationship between the number of originally given basis elements (n) and the number of higher-dimensional elements of dimension d that can be constructed from the original elements; this number is $\binom{n}{d}$ and is shown in the Pascal's triangle illustration next to the respective sets. For example, b_1, b_2 and b_3 are the one-dimensional basis elements for $n=3$. There are $\binom{3}{2}=3$ two-dimensional basis elements one can construct, called b_{12}, b_{13} and b_{23}. Further, $\binom{3}{3}=1$ three-dimensional basis element can be defined, called b_{123}. (The set $\{\ \}$ is shown for $\binom{n}{0}$ merely for "completeness." One can view it as a representation of a zero-dimensional element.)



The left figure concerns the issue of "sign and orientation." Since each basis element b_i can be used as $+b_i$ or $-b_i$, there are 2^4 "signed sets" for $n=4$ and $d=1$: $\{+b_1, +b_2, +b_3, +b_4\}, \ldots,$ ...

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

- Laplacian eigenfunctions and neural networks:... $\ldots, \{-b_1, -b_2, -b_3, -b_4\}$. Since each basis element $b_{i_1 i_2 i_3}$ in the basis element set $\{b_{123}, b_{124}, b_{134}, b_{234}\}$, where $n=4$ and $d=3$, has four elements, there are 2^4 "signed sets," i.e., $\{+b_{123}, +b_{124}, +b_{134}, +b_{234}\}$, $\ldots$, $\{-b_{123}, -b_{124}, -b_{134}, -b_{234}\}$. Analogously, there are $2^6 = 64$ signed basis element sets for $n=4$ and $d=2$. Generally, there exist $2^{\binom{n}{d}}$ signed basis element sets. (While this discussion is rather abstract and purely uses combinatorics, the relationship to signs, orientations and generalized "basis elements of different dimensions" will become apparent in the following discussion.)

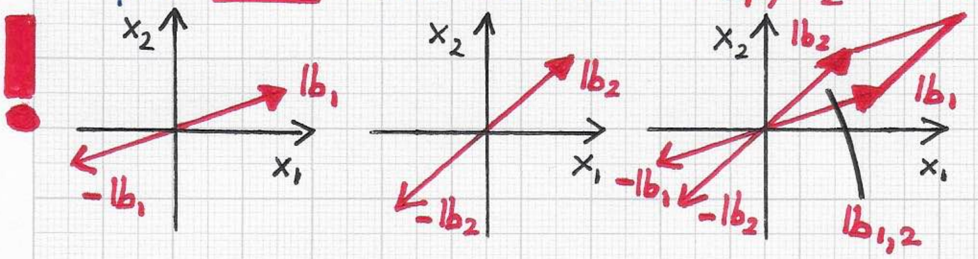
(• Two references regarding quaternions and geometric computing are:
  - Andrew J. Hanson, Visualizing Quaternions, Morgan Kaufmann, 2006.
  - Geometric Computing with Clifford Algebras, Gerald Sommer, editor, Springer, 2001.

  A reference for geometric algebra is:
  - Leo Dorst, Daniel Fontijne and Stephen Mann, Geometric Algebra for Computer Science, revised edition, Morgan Kaufmann, 2009.)

...

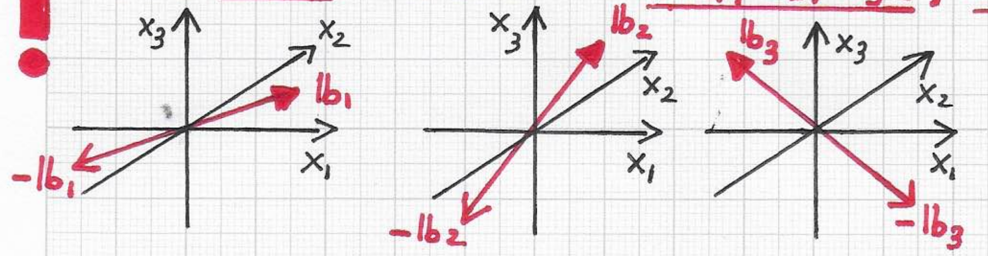
OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:… In our context, one can view the "basis elements" $b_1, b_2, \ldots, b_n$ as vectors, i.e., $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n$. For example, for $n=2$ one obtains $\{\mathbf{b}_1, \mathbf{b}_2\}$ and $\{\mathbf{b}_{1,2}\}$. The

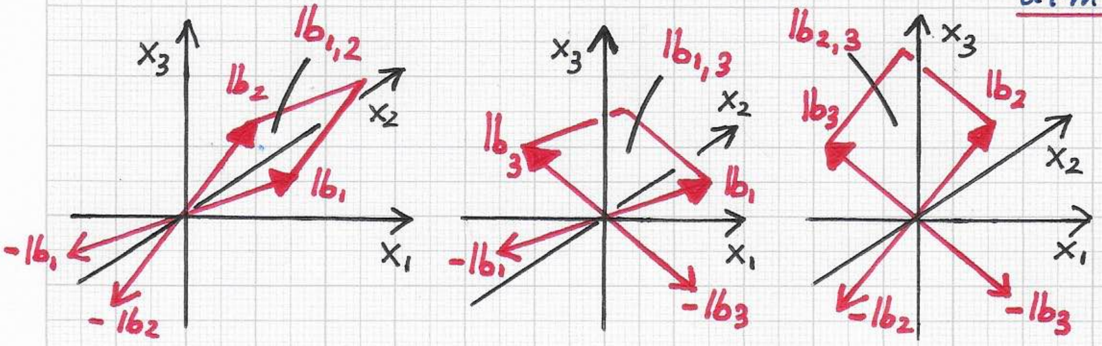


Left figures show all possible ways how the given two vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ can be used to define bases for one- and two-dimensional spaces. (The parallelogram spanned by $\mathbf{b}_1$ and $\mathbf{b}_2$ is called $\mathbf{b}_{1,2}$ and can be understood as a two-dimensional "basis element.")

For $n=3$ one obtains $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$, $\{\mathbf{b}_{1,2}, \mathbf{b}_{1,3}, \mathbf{b}_{2,3}\}$ and



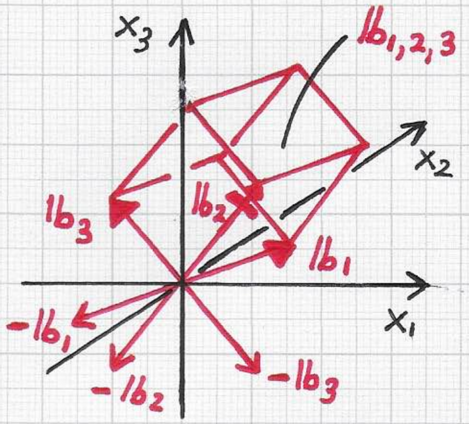
$\{\mathbf{b}_{1,2,3}\}$. The left figures show the one- and two-dimensional "basis elements."



The vectors with negative sign, i.e., $-\mathbf{b}_1$, $-\mathbf{b}_2$ and $-\mathbf{b}_3$, are shown for context: The various possibilities for the two-dimensional "basis elements" are obtained by combining vectors with '+' and '−' signs to define parallelograms. …

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:...



The three-dimenensional "basis element" $\mathbb{b}_{1,2,3}$ (parallelepiped) is shown in the left figure. In this case, one can define eight parallelepipeds — by using the sign-based possibilities $+\mathbb{b}_1, +\mathbb{b}_2, +\mathbb{b}_3$; ... ...; and $-\mathbb{b}_1, -\mathbb{b}_2, -\mathbb{b}_3$. These eight triples of signed vectors generate (span) parallelepipeds with the same absolute volume. Nevertheless, the volume itself can be positive or negative. One could refer to such "basis elements" as "**UNI**-element" (b_i), "**BI**-element" ($b_{i,j}$) or "**TRI**-element" ($b_{i,j,k}$) to indicate a 1D, 2D or 3D case.

• Geometric algebra provides a foundation for applying geometric operations "directly" to geometric objects to be manipulated. The GEOMETRIC PRODUCT is the enabling operation for this purpose. This product can be decomposed into the INNER (scalar, dot) PRODUCT and the OUTER (exterior, wedge) PRODUCT. In the following, we write the inner product as $\langle \mathbb{u}, \mathbb{v} \rangle$, the outer product as $\mathbb{u} \wedge \mathbb{v}$ and scalar multiplication of a vector as $\lambda \mathbb{u}$ or $\lambda \cdot \mathbb{u}$.

...