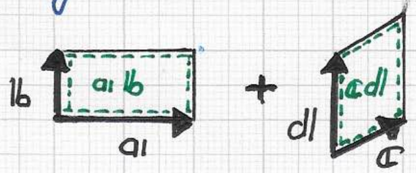
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

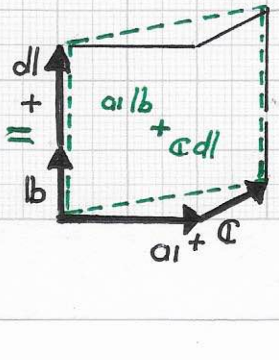
• Laplacian eigenfunctions and neural networks:... (Geometric algebra is covered in many books and papers. We refer to the following publications for details:

– Alan MacDonald, A Survey of Geometric Algebra and Geometric Calculus, Adv. Appl. Cliff. Alg. 27, pp. 853–891, 2017.

– Anthony D. DeRose, A Coordinate-free Approach to Geometric Programming, in: Wolfgang Straßer and Hans-Peter Seidel, eds., Theory and Practice of Geometric Modeling, Springer-Verlag, pp. 291–305, 1989.)

Geometric algebra can be viewed as a generalization of "standard vector algebra." Combining vectors can be done to define parallelograms; combining vectors and parallelograms can be done to define parallelepipeds; combining these geometrical objects can be done to define high-dimensional "hyper-parallelepipeds." These operations can be generalized for n-dimensional space. The simple example shown in the left figure for $n=2$ illustrates how one can define an "addition of two parallelograms": The vector pair $\mathbf{a}, \mathbf{b}$ spans the parallelogram $\mathbf{a}\mathbf{b}$, and the pair $\mathbf{c}, \mathbf{d}$ spans $\mathbf{c}\mathbf{d}$. The sum $\mathbf{a}\mathbf{b} + \mathbf{c}\mathbf{d}$ is the parallelogram spanned by the pair $\mathbf{a}+\mathbf{c}$, $\mathbf{b}+\mathbf{d}$. ...

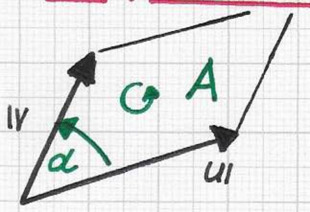




OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

- **Laplacian eigenfunctions and neural networks:...** More formally, one can write this sum as $(\mathbf{a}|\mathbf{b}) + (\mathbf{c}|\mathbf{d}) = (\mathbf{a}+\mathbf{c})(\mathbf{b}+\mathbf{d})$.

- **Definitions, rules, properties and conventions.**



The left figure and associated equations point out some relevant aspects of the geometric product, for two vectors $\mathbf{u} = (u_1, u_2)^T$ and $\mathbf{v} = (v_1, v_2)^T$ and $n = 2$. One can decompose the 2-by-2 matrix resulting from $\mathbf{u}\mathbf{v}^T$ into a "symmetric" and "anti-symmetric" matrix; the geometric product generally has a symmetric and anti-symmetric part. One uses the following product decomposition:

$$\mathbf{u}\mathbf{v}^T = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} (v_1, v_2) = \begin{pmatrix} u_1 v_1 & u_1 v_2 \\ u_2 v_1 & u_2 v_2 \end{pmatrix}$$

$$= \begin{pmatrix} u_1 v_1 & 0 \\ 0 & u_2 v_2 \end{pmatrix} + \begin{pmatrix} 0 & u_1 v_2 \\ u_2 v_1 & 0 \end{pmatrix} = \text{"symm.} + \text{anti-symm."}$$

$$\cos\alpha = \langle \mathbf{u}, \mathbf{v} \rangle / (\|\mathbf{u}\| \|\mathbf{v}\|)$$

$$i \cdot \sin\alpha = (\mathbf{u} \wedge \mathbf{v}) / (\|\mathbf{u}\| \|\mathbf{v}\|)$$

$$A = \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} = u_1 v_2 - u_2 v_1$$

$$\mathbf{u}\mathbf{v} = \tfrac{1}{2}(\mathbf{u}\mathbf{v} + \mathbf{v}\mathbf{u}) + \tfrac{1}{2}(\mathbf{u}\mathbf{v} - \mathbf{v}\mathbf{u}) =$$

$$= \langle \mathbf{u}, \mathbf{v} \rangle + (\mathbf{u} \wedge \mathbf{v}) =$$

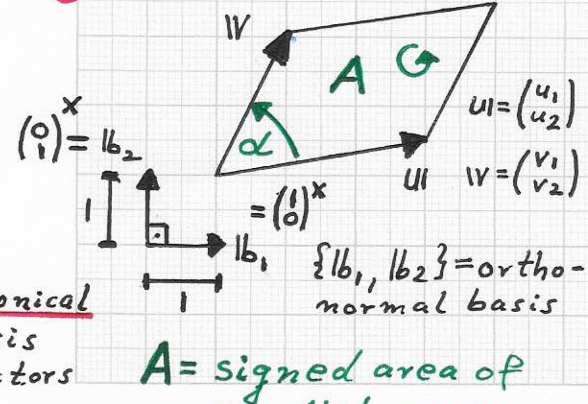
$$= (u_1 v_1 + u_2 v_2) + (u_1 v_2 - u_2 v_1)\, \mathbf{b}_1 \mathbf{b}_2$$

$$= \|\mathbf{u}\| \|\mathbf{v}\| \cos\alpha + A\, i =$$

(We assume that $\{\mathbf{b}_1, \mathbf{b}_2\}$ is an orthonormal basis of the plane, i.e., $\mathbf{u} = u_1 \mathbf{b}_1 + u_2 \mathbf{b}_2$ and $\mathbf{v} = v_1 \mathbf{b}_1 + v_2 \mathbf{b}_2$.)

$$= \|\mathbf{u}\| \|\mathbf{v}\| \cos\alpha + i \|\mathbf{u}\| \|\mathbf{v}\| \sin\alpha \, .$$

Here, $\mathbf{b}_1 \mathbf{b}_2 = i$ defines the parallelogram plane. ...



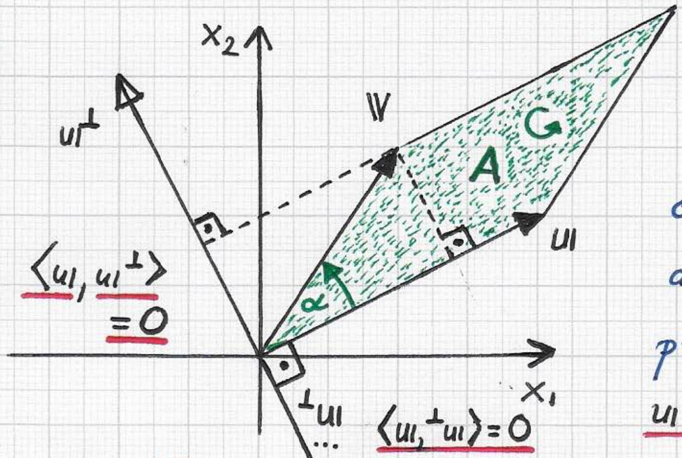
$\{\mathbf{b}_1, \mathbf{b}_2\}$ = orthonormal basis

^{x}canonical basis vectors

A = signed area of parallelogram

■ OBJECT AND MATERIAL EIGENFUNCTIONS - Cont'd.

• Laplacian eigenfunctions and neural networks:...



• By applying the cos and sin rules to the simple geometrical example shown in the left figure ($n=2$), one can explain the terms appearing in the geometric product of two vectors. Here, $\mathbb{u} = (u_1, u_2)^T$, $\mathbb{u}^\perp = (-u_2, u_1)^T$, $\mathbb{v} = (v_1, v_2)^T$.

For the oriented angle α, one obtains the equations

$$c\alpha = \cos\alpha = \langle \mathbb{v}, \mathbb{u}/\|\mathbb{u}\| \rangle / \|\mathbb{v}\| = \langle \mathbb{u}, \mathbb{v} \rangle / (\|\mathbb{u}\| \, \|\mathbb{v}\|) \;,$$

$$s\alpha = \sin\alpha = \langle \mathbb{v}, \mathbb{u}^\perp/\|\mathbb{u}^\perp\| \rangle / \|\mathbb{v}\| = \langle \mathbb{u}^\perp, \mathbb{v} \rangle / (\|\mathbb{u}\| \, \|\mathbb{v}\|) \;.$$

Thus, one has the results

$$\langle \mathbb{u}, \mathbb{v} \rangle = \|\mathbb{u}\| \, \|\mathbb{v}\| \cos\alpha \;, \quad \langle \mathbb{u}^\perp, \mathbb{v} \rangle = \|\mathbb{u}\| \, \|\mathbb{v}\| \sin\alpha \;,$$

which are the expressions appearing on the previous page in the formula for the geometric product $\mathbb{u}\mathbb{v}$. Further, the value of $\langle \mathbb{u}^\perp, \mathbb{v} \rangle = (u_1 v_2 - u_2 v_1)$ is the value of the signed area **A** shown in the top-left figure and the figure at the bottom of the previous page. Thus, $\mathbb{u}\mathbb{v} = \langle \mathbb{u}, \mathbb{v} \rangle + i\mathbf{A} = \langle \mathbb{u}, \mathbb{v} \rangle + i \langle \mathbb{u}^\perp, \mathbb{v} \rangle$ when adopting complex number notation.

(The vector ${}^\perp\mathbb{u} = (u_2, -u_1)^T$ is the second vector that is orthogonal to $\mathbb{u}$: ${}^\perp\mathbb{u} = -\mathbb{u}^\perp \wedge \langle \mathbb{u}, \mathbb{u}^\perp \rangle = 0 \Rightarrow$ $\langle \mathbb{u}, {}^\perp\mathbb{u} \rangle = \langle \mathbb{u}, -\mathbb{u}^\perp \rangle = -\langle \mathbb{u}, \mathbb{u}^\perp \rangle = 0 \Rightarrow \langle {}^\perp\mathbb{u}, \mathbb{v} \rangle =$ $u_2 v_1 - u_1 v_2 = -(u_1 v_2 - u_2 v_1) = -\mathbf{A}$. Thus, changing the sign/orientation of $\mathbb{u}^\perp$ changes orientation and sign of the area.)

...

OBJECT AND MATERIAL EIGENFUNCTIONS - Cont'd.

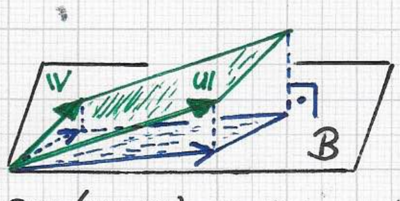
- Laplacian eigenfunctions and neural networks:... • It should also be noted that the inner product of two vectors $\langle \mathbf{u}, \mathbf{v} \rangle$, i.e., the symmetric part of the geometric product, implies that $\mathbf{u}^2 = \frac{1}{2}(\mathbf{u}\mathbf{u} + \mathbf{u}\mathbf{u}) = \langle \mathbf{u}, \mathbf{u} \rangle = \|\mathbf{u}\|^2$. Thus, the vector $\mathbf{u}$, $\mathbf{u} \neq \mathbf{0}$, has the inverse $\mathbf{u}^{-1} = \mathbf{u} / \|\mathbf{u}\|^2$.
  If $\langle \mathbf{u}, \mathbf{v} \rangle = 0$ ($\mathbf{u}$ and $\mathbf{v}$ being orthogonal), then only the anti-symmetric part of the geometric product is different from 0, and one will obtain $\mathbf{u}\mathbf{v} = 0 + \frac{1}{2}(\mathbf{u}\mathbf{v} - \mathbf{v}\mathbf{u}) \Rightarrow \mathbf{u}\mathbf{v} = -\mathbf{v}\mathbf{u}$.
- As before, we call the normalized and mutually-orthogonal vectors defining an orthonormal basis $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n$ (n-dimensional space). The basis $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ — where one can view the indices of the basis vectors defining an order for the basis vectors $(1 < 2 < \ldots < n)$ — implies the possible **k-vectors** defining the "basis elements" of the geometric algebra. The left figure includes all possible k-vectors for $n=4$. The k-vectors are listed in the rows for $k = 0, 1, 2, 3$ and 4. They are also called scalar ($k=0$), vectors ($k=1$), bivectors ($k=2$), trivectors ($k=3$) and quadrivector ($k=4$), respectively. ...

	$\binom{n}{k}$	$n=4$
$k=0$	1	1
$=1$	4	$\mathbf{b}_1$ $\mathbf{b}_2$ $\mathbf{b}_3$ $\mathbf{b}_4$
$=2$	6	$\mathbf{b}_1\mathbf{b}_2$ $\mathbf{b}_1\mathbf{b}_3$ $\mathbf{b}_1\mathbf{b}_4$ $\mathbf{b}_2\mathbf{b}_3$ $\mathbf{b}_2\mathbf{b}_4$ $\mathbf{b}_3\mathbf{b}_4$
$=3$	4	$\mathbf{b}_1\mathbf{b}_2\mathbf{b}_3$ $\mathbf{b}_1\mathbf{b}_2\mathbf{b}_4$ $\mathbf{b}_1\mathbf{b}_3\mathbf{b}_4$ $\mathbf{b}_2\mathbf{b}_3\mathbf{b}_4$
$=4$	1	$\mathbf{b}_1\mathbf{b}_2\mathbf{b}_3\mathbf{b}_4$

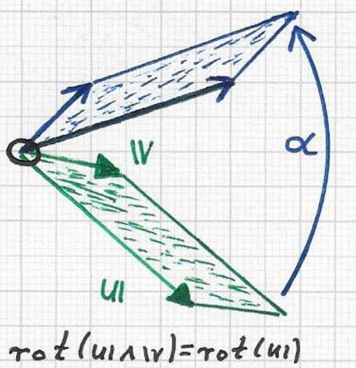
$\Rightarrow 2^n = 2^4 = 16$ "basis elements"

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

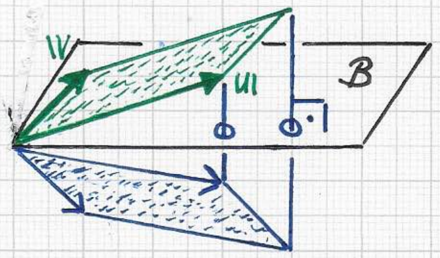
• Laplacian eigenfunctions and neural networks:… • OBJECTS are represented by combinations of k-vectors. They are called BLADES. OPERATIONS for objects are performed by applying operations to blades. For the general n-dimensional case, and a specific value k $(0 \le k \le n)$, there exist $\binom{n}{k}$ possibilities to define a BLADE as a PRODUCT of orthonormal basis vectors $\mathbb{b}_{i_1}, \mathbb{b}_{i_2}, \ldots, \mathbb{b}_{i_k}$, with $i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, n\}$ and all indices i_l different from each other. This blade is also more precisely called k-BLADE. Thus, a k-blade defines a k-dimensional subspace in n-dimensional space. Changing the sign of a k-blade changes the k-blade's orientation. Of particular interest are the "standard" linear mappings (in n-dimensional space). The left figure provides abstract visualizations of a projection (pro), a reflection (ref) and a rotation (rot) relative to a k-blade $\mathcal{B}$. Performing rotations in n-dimensional space via geometric algebra is especially relevant and advantageous. …



$pro(\mathbb{u} \wedge \mathbb{v}) = pro(\mathbb{u}) \wedge pro(\mathbb{v})$



$rot(\mathbb{u} \wedge \mathbb{v}) = rot(\mathbb{u}) \wedge rot(\mathbb{v})$



$ref(\mathbb{u} \wedge \mathbb{v}) = ref(\mathbb{u}) \wedge ref(\mathbb{v})$

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