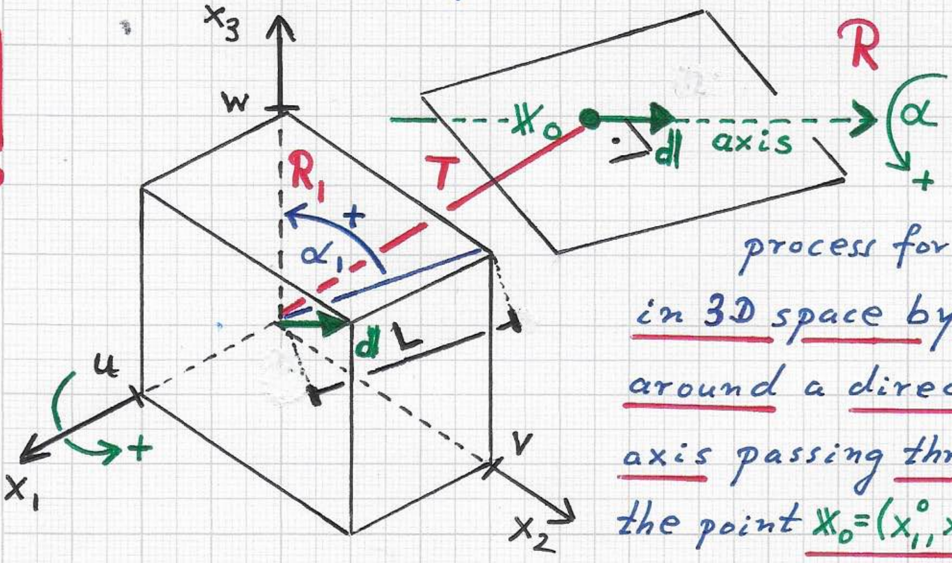
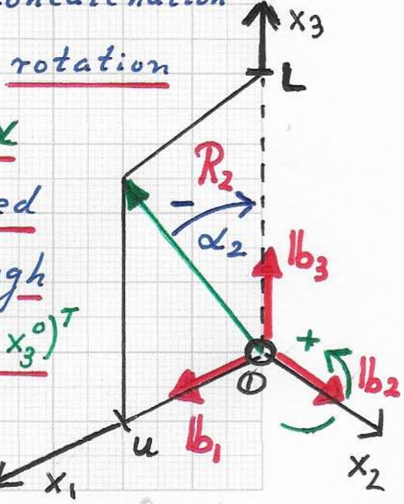
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... • We briefly review the "simplification" of an arbitrary rotation in 3D space by decomposing the desired rotation operation into "basic transformations" that are concatenated – as commonly done in computer graphics. The main underlying fact used in this context is the fact that a rotation is always an operation "taking place in a plane, i.e., a 2D subspace," regardless of the dimension of the n-dimensional embedding space ($n \geq 2$). Understanding a general rotation as a concatenation of "basic transformations" is a more intuitive approach – and the final, complicated matrix product must be computed only once.



The figures on this page show a concatenation process for a rotation in 3D space by α around a directed axis passing through the point $\mathbf{x}_0 = (x_1^0, x_2^0, x_3^0)^T$ and having the unit direction vector $\mathbf{d} = (u, v, w)^T$, $\|\mathbf{d}\| = 1$. This general rotation is "reduced" to a rotation in the x_1x_2-plane. ...



OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:… The general rotation R is decomposed into the following sequence of "basic transformations":

i) T: translation by positional vector $-\mathbf{x}_0$
ii) R_1: rotation by α_1 around x_1-axis (+ rotation)
iii) R_2: rotation by $-\alpha_2$ around x_2-axis (– rotation)
iv) R_3: rotation by α around x_3-axis (+ rotation)
v) R_2^{-1}: inverse of rotation R_2
vi) R_1^{-1}: inverse of rotation R_1
vii) T^{-1}: inverse of translation T

If we view the names of these seven transformations as matrices, then the final general rotation matrix R will be $T_1^{-1} R_1^{-1} R_2^{-1} R_3 R_2 R_1 T$. It is assumed that all matrices are written in homogeneous form, i.e., as 4-by-4 matrices:

$$T=\begin{bmatrix}1&0&0&-x_1^0\\0&1&0&-x_2^0\\0&0&1&-x_3^0\\0&0&0&1\end{bmatrix},\quad T^{-1}=\begin{bmatrix}1&0&0&x_1^0\\0&1&0&x_2^0\\0&0&1&x_3^0\\0&0&0&1\end{bmatrix},\quad R_1=\begin{bmatrix}1&0&0&0\\0&c\alpha_1&-s\alpha_1&0\\0&s\alpha_1&c\alpha_1&0\\0&0&0&1\end{bmatrix},\quad R_1^{-1}=R_1^T,$$

$$R_2=\begin{bmatrix}c(-\alpha_2)&0&+s(-\alpha_2)&0\\0&1&0&0\\-s(-\alpha_2)&0&c(-\alpha_2)&0\\0&0&0&1\end{bmatrix}=\begin{bmatrix}c\alpha_2&0&-s\alpha_2&0\\0&1&0&0\\s\alpha_2&0&c\alpha_2&0\\0&0&0&1\end{bmatrix},\quad R_2^{-1}=R_2^T,$$

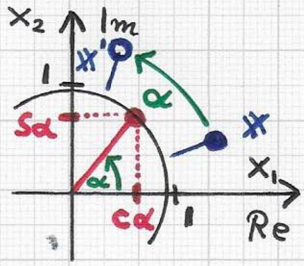
$$R_3=\begin{bmatrix}c\alpha&-s\alpha&0&0\\s\alpha&c\alpha&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}.$$

The figures on the previous page provide the needed values for α_1 and α_2: $c\alpha_1=\cos\alpha_1=w/L$, $s\alpha_1=\sin\alpha_1=v/L$; $c\alpha_2=\cos\alpha_2=L/1=L$, $s\alpha_2=\sin\alpha_2=u/1=u$; where $L=(v^2+w^2)^{1/2}$.

…

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... The orthonormal basis vectors defining the 3D coordinate system (together with its origin $\mathbb{O}$) are $\mathbb{b}_1$, $\mathbb{b}_2$ and $\mathbb{b}_3$. The described concatenation method is based on the idea of using the plane spanned by vectors $\mathbb{b}_1$ and $\mathbb{b}_2$ as the plane (= 2D subspace) for executing the rotation, using the x_3-axis with direction $\mathbb{b}_3$ as rotation axis. There exists a close relationship between this idea and using geometric algebra for performing general rotations in an n-dimensional space.



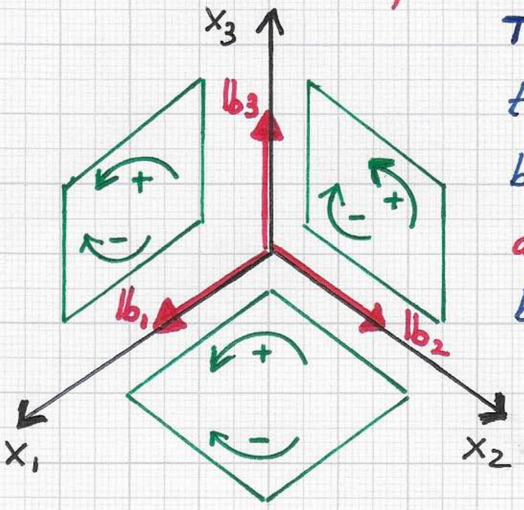
• Another relationship exists between complex numbers and geometric algebra in the context of rotations. The left figure shows how complex numbers can be used to describe rotations in the (complex) plane. The desired rotation "operator" is the complex number $x_R = c\alpha + i s\alpha$; the point/number to be rotated is $x = x_1 + i x_2$ ($\hat{=} (x_1, x_2)^T$); the image of x is $x' = x_R x = (c\alpha + i s\alpha)(x_1 + i x_2) = (x_1 c\alpha - x_2 s\alpha) + i(x_1 s\alpha + x_2 c\alpha)$. This result is equivalent to the result $\begin{pmatrix} c\alpha & -s\alpha \\ s\alpha & c\alpha \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 c\alpha - x_2 s\alpha \\ x_1 s\alpha + x_2 c\alpha \end{pmatrix}$.

In the following, these relationships will become clearer. We will discuss the wedge product, multiplication rules and rotors. ...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks: ...

• We will now focus on geometric algebra and the wedge product, to see the close relationships with the described related topics. The goal is an "intuitive understanding" of a ROTOR, a rotation operation that directly applies to n-dimensional space.

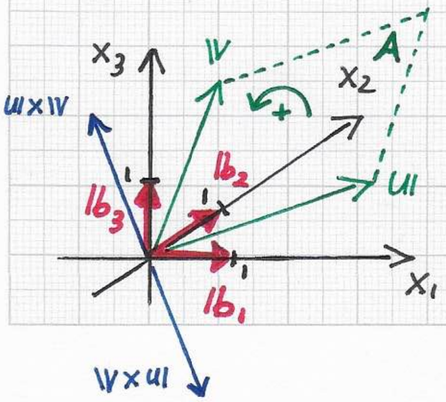


The left figure and multiplication table summarize the relationships between basis vectors $\mathbb{b}_1, \mathbb{b}_2, \mathbb{b}_3$ and basis bivectors $\mathbb{b}_{12}, \mathbb{b}_{23}, \mathbb{b}_{31}$; between wedge products and the associated parallelograms; and between orientation and sign of wedge products. Anti-commutativity yields the wedge products

$$\mathbb{b}_{12} = \mathbb{b}_1 \wedge \mathbb{b}_2 = -\mathbb{b}_{21} \quad ,$$
$$\mathbb{b}_{23} = \mathbb{b}_2 \wedge \mathbb{b}_3 = -\mathbb{b}_{32} \quad ,$$
$$\mathbb{b}_{31} = \mathbb{b}_3 \wedge \mathbb{b}_1 = -\mathbb{b}_{13} \quad .$$

$\wedge$	$\mathbb{b}_1$	$\mathbb{b}_2$	$\mathbb{b}_3$
$\mathbb{b}_1$	0	$\mathbb{b}_{12}$	$-\mathbb{b}_{31}$
$\mathbb{b}_2$	$-\mathbb{b}_{12}$	0	$\mathbb{b}_{23}$
$\mathbb{b}_3$	$\mathbb{b}_{31}$	$-\mathbb{b}_{23}$	0

$$(\mathbb{u} \wedge \mathbb{v}) = -(\mathbb{v} \wedge \mathbb{u})$$



These rules can now be used to compute the wedge product of two vectors $\mathbb{u} = (u_1, u_2, u_3)^T$ and $\mathbb{v} = (v_1, v_2, v_3)^T$:

$$\mathbb{u} \wedge \mathbb{v} = (u_1 \mathbb{b}_1 + u_2 \mathbb{b}_2 + u_3 \mathbb{b}_3) \wedge (v_1 \mathbb{b}_1 + v_2 \mathbb{b}_2 + v_3 \mathbb{b}_3) =$$
$$= u_1 v_1 \mathbb{b}_1^2 + u_1 v_2 \mathbb{b}_1 \mathbb{b}_2 + u_1 v_3 \mathbb{b}_1 \mathbb{b}_3 + u_2 v_1 \mathbb{b}_2 \mathbb{b}_1 + u_2 v_2 \mathbb{b}_2^2 + u_2 v_3 \mathbb{b}_2 \mathbb{b}_3$$
$$+ u_3 v_1 \mathbb{b}_3 \mathbb{b}_1 + u_3 v_2 \mathbb{b}_3 \mathbb{b}_2 + u_3 v_3 \mathbb{b}_3^2 = (u_1 v_2 - u_2 v_1)\, \mathbb{b}_{12} + \ldots$$
$$+ (-u_1 v_3 + u_3 v_1)\, \mathbb{b}_{31} + (u_2 v_3 - u_3 v_2)\, \mathbb{b}_{23} \, .$$

...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... The figure at the bottom of the previous page also includes the cross products $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$. The cross product $\mathbf{u} \times \mathbf{v}$ is

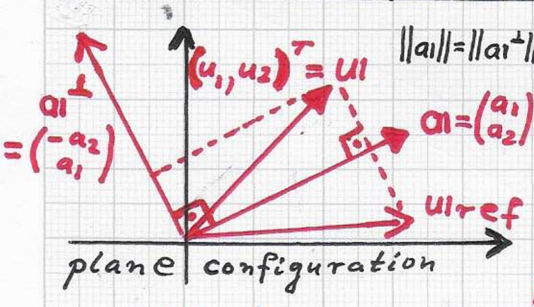
$$\begin{vmatrix} u_1 & v_1 & i \\ u_2 & v_2 & j \\ u_3 & v_3 & k \end{vmatrix} = \begin{matrix} i\,(u_2 v_3 - u_3 v_2) \\ +j\,(-u_1 v_3 + u_3 v_1) \\ +k\,(u_1 v_2 - u_2 v_1) \end{matrix} \hat{=} \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ -u_1 v_3 + u_3 v_1 \\ u_1 v_2 - u_2 v_1 \end{bmatrix} = \mathbf{u} \times \mathbf{v}.$$

Thus, $\|\mathbf{u} \wedge \mathbf{v}\| = \|\mathbf{u} \times \mathbf{v}\| = |\text{area}|$ of parallelogram (A).

• We point out again that $\mathbf{u}\mathbf{v} = \langle \mathbf{u}, \mathbf{v} \rangle + (\mathbf{u} \wedge \mathbf{v})$, implying that the orthonormal basis vectors satisfy $\mathbf{b}_i^2 = \langle \mathbf{b}_i, \mathbf{b}_i \rangle = 1$ and $\mathbf{b}_i \mathbf{b}_j = 0 + (\mathbf{b}_i \wedge \mathbf{b}_j) = (\mathbf{b}_i \wedge \mathbf{b}_j) = \mathbf{b}_{ij} = -\mathbf{b}_{ji}$, $i \neq j$.

$\wedge$	$\mathbf{b}_1$	$\mathbf{b}_2$	$\mathbf{b}_3$	$\mathbf{b}_{12}$	$\mathbf{b}_{23}$	$\mathbf{b}_{31}$	$\mathbf{b}_{123}$
$\mathbf{b}_1$	1	$\mathbf{b}_{12}$	$-\mathbf{b}_{31}$	$\mathbf{b}_2$	$\mathbf{b}_{123}$	$-\mathbf{b}_3$	$\mathbf{b}_{23}$
$\mathbf{b}_2$	$-\mathbf{b}_{12}$	1	$\mathbf{b}_{23}$	$-\mathbf{b}_1$	$\mathbf{b}_3$	$\mathbf{b}_{123}$	$\mathbf{b}_{31}$
$\mathbf{b}_3$	$\mathbf{b}_{31}$	$-\mathbf{b}_{23}$	1	$\mathbf{b}_{123}$	$-\mathbf{b}_2$	$\mathbf{b}_1$	$\mathbf{b}_{12}$
$\mathbf{b}_{12}$	$-\mathbf{b}_2$	$\mathbf{b}_1$	$\mathbf{b}_{123}$	-1	$-\mathbf{b}_{31}$	$\mathbf{b}_{23}$	$-\mathbf{b}_3$
$\mathbf{b}_{23}$	$\mathbf{b}_{123}$	$-\mathbf{b}_3$	$\mathbf{b}_2$	$\mathbf{b}_{31}$	-1	$-\mathbf{b}_{12}$	$-\mathbf{b}_1$
$\mathbf{b}_{31}$	$\mathbf{b}_3$	$\mathbf{b}_{123}$	$-\mathbf{b}_1$	$-\mathbf{b}_{23}$	$\mathbf{b}_{12}$	-1	$-\mathbf{b}_2$
$\mathbf{b}_{123}$	$\mathbf{b}_{23}$	$\mathbf{b}_{31}$	$\mathbf{b}_{12}$	$-\mathbf{b}_3$	$-\mathbf{b}_1$	$-\mathbf{b}_2$	-1
$\mathbf{b}_4$	$-\mathbf{b}_{14}$	$-\mathbf{b}_{24}$	...	...			...
$\mathbf{b}_5$	$-\mathbf{b}_{15}$	$-\mathbf{b}_{25}$	...			...	...

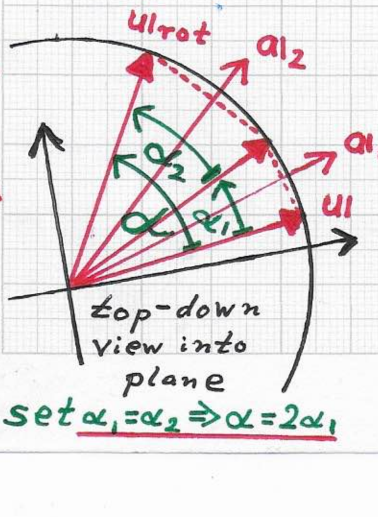
The left table lists the possible wedge products for $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{31}, \mathbf{b}_{123}$ (and indicates how to extend the table). The left figure shows the elementary operation of a **ROTOR**, the "perpendicular reflection" of $\mathbf{u}$ relative to axis vector $\mathbf{a}$:



$$\mathbf{u}_{ref} = \mathbf{u} - 2\,\frac{\langle \mathbf{u}, \mathbf{a}^\perp \rangle}{\|\mathbf{a}\|^2} \cdot \mathbf{a}^\perp = \ldots = \mathbf{a}\,\mathbf{u}\,\mathbf{a}^{-1}.$$

• As discussed earlier, we can achieve a rotation by an angle via two reflections, see left figure. IN GEOMETRIC ALGEBRA, THIS ROTATION IS EXPRESSED AS $\mathbf{u}_{rot} = \mathbf{a}_2\, \mathbf{a}_1\, \mathbf{u}\, \mathbf{a}_1^{-1}\, \mathbf{a}_2^{-1} = R\, \mathbf{u}\, R^{-1}$. THE **ROTOR** IS CALLED R. •••

$R = \mathbf{a}_2 \mathbf{a}_1 = \ldots = \|\mathbf{a}_2\| \|\mathbf{a}_1\| \cos\frac{\alpha}{2} + \|\mathbf{a}_2\| \|\mathbf{a}_1\| \sin\frac{\alpha}{2} \cdot \left(\frac{\mathbf{a}_2 \wedge \mathbf{a}_1}{\|\mathbf{a}_2 \wedge \mathbf{a}_1\|} \right)$



set $\alpha_1 = \alpha_2 \Rightarrow \alpha = 2\alpha_1$