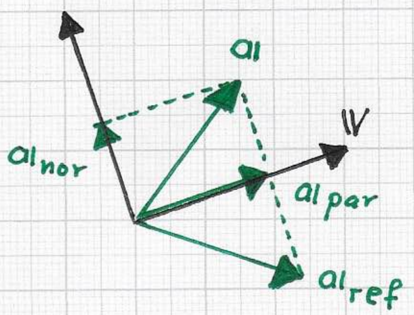
Stratovan

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks: ...

• **Note.** The introduction to and description of a **ROTOR** on the previous pages is rather "compact and condensed." A more detailed presentation of the most relevant computations using the rules of **GEOMETRIC ALGEBRA** should be helpful. **First,** we recall that a vector can be written as the sum of a "parallel" (par) and "normal" (nor) vector, see left figure:



$$\mathbf{a} = \mathbf{a}_{par} + \mathbf{a}_{nor} = \langle \mathbf{a}, \mathbf{v}\rangle \frac{\mathbf{v}}{\|\mathbf{v}\|^2} + (\mathbf{a}\wedge\mathbf{v})\frac{\mathbf{v}}{\|\mathbf{v}\|^2}.$$

Second, we can directly derive the formula for the "reflection" (ref) of $\mathbf{a}$ relative to the reflection axis defined by $\mathbf{v}$:

$$\begin{aligned}
\mathbf{a}_{ref} &= \mathbf{a}_{par} - \mathbf{a}_{nor} = \langle \mathbf{a}, \mathbf{v}\rangle \frac{\mathbf{v}}{\|\mathbf{v}\|^2} - (\mathbf{a}\wedge\mathbf{v})\frac{\mathbf{v}}{\|\mathbf{v}\|^2} = \\
&= \langle \mathbf{a}, \mathbf{v}\rangle \mathbf{v}^{-1} - (\mathbf{a}\wedge\mathbf{v})\mathbf{v}^{-1} = \quad (\text{using } \mathbf{v}^{-1} = \mathbf{v}/\|\mathbf{v}\|^2) \\
&= (\langle \mathbf{a}, \mathbf{v}\rangle + (\mathbf{v}\wedge\mathbf{a}))\,\mathbf{v}^{-1} = \\
&= (\langle \mathbf{v}, \mathbf{a}\rangle + (\mathbf{v}\wedge\mathbf{a}))\,\mathbf{v}^{-1} = (\mathbf{v}\mathbf{a})\,\mathbf{v}^{-1} = \mathbf{v}\,\mathbf{a}\,\mathbf{v}^{-1}.
\end{aligned}$$

Given: $\mathbf{v}$ and $\mathbf{w}$

Third, as pointed out and discussed before, a rotation can be achieved by performing two reflections in sequence. Thus, by performing a reflection to $\mathbf{a}_{ref}$ we obtain

$$\begin{aligned}
\mathbf{a}_{rot} &= \mathbf{w}\,\mathbf{a}_{ref}\,\mathbf{w}^{-1} = \mathbf{w}\,(\mathbf{v}\,\mathbf{a}\,\mathbf{v}^{-1})\,\mathbf{w}^{-1} = \\
&= \mathbf{w}\,\mathbf{v}\,\mathbf{a}\,\mathbf{v}^{-1}\mathbf{w}^{-1} = \mathbf{w}\,\mathbf{v}\,\mathbf{a}\,(\mathbf{w}\,\mathbf{v})^{-1} = \quad (\text{using } \mathbf{v}^{-1}\mathbf{w}^{-1} = (\mathbf{w}\mathbf{v})^{-1}) \\
&= (\mathbf{w}\,\mathbf{v})\,\mathbf{a}\,(\mathbf{w}\,\mathbf{v})^{-1} = R\,\mathbf{a}\,R^{-1}.
\end{aligned}$$

This formal derivation defines a **ROTOR R**. ...

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

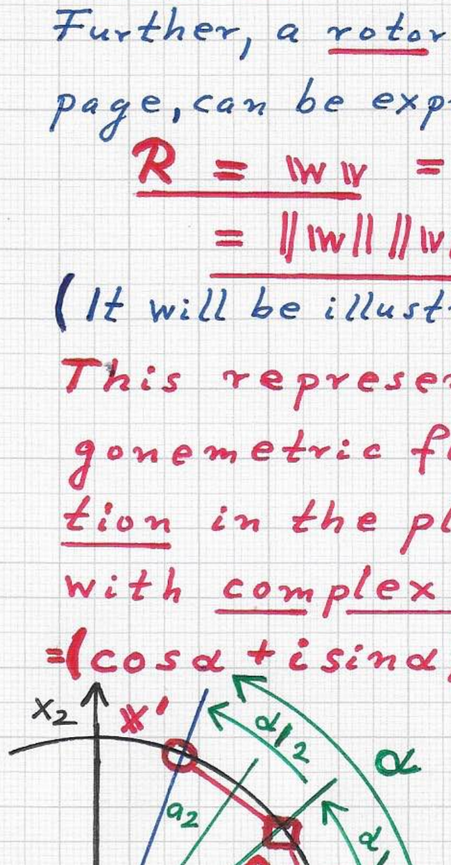
• Laplacian eigenfunctions and neural networks:... It must be emphasized that the simple 2D and 3D illustrations used to clarify the concepts and rules of geometric algebra merely represent low-dimensional simplifications of the underlying n-dimensional theory and its application. Nevertheless, one can and must keep in mind that a rotation is a mapping that takes place in a plane, a 2D subspace.

Further, a rotor, as formally defined on the previous page, can be expressed via the cos and sin functions:

$$\underline{R = \mathbb{w}\mathbb{v}} = \langle \mathbb{w}, \mathbb{v} \rangle + (\mathbb{w} \wedge \mathbb{v}) =$$
$$= \|\mathbb{w}\|\,\|\mathbb{v}\| \cos\tfrac{\alpha}{2} + \|\mathbb{w}\|\,\|\mathbb{v}\| \sin\tfrac{\alpha}{2} \frac{\mathbb{w} \wedge \mathbb{v}}{|\mathbb{w} \wedge \mathbb{v}|}.$$

(It will be illustrated in a figure why $\frac{\alpha}{2}$ is used.)

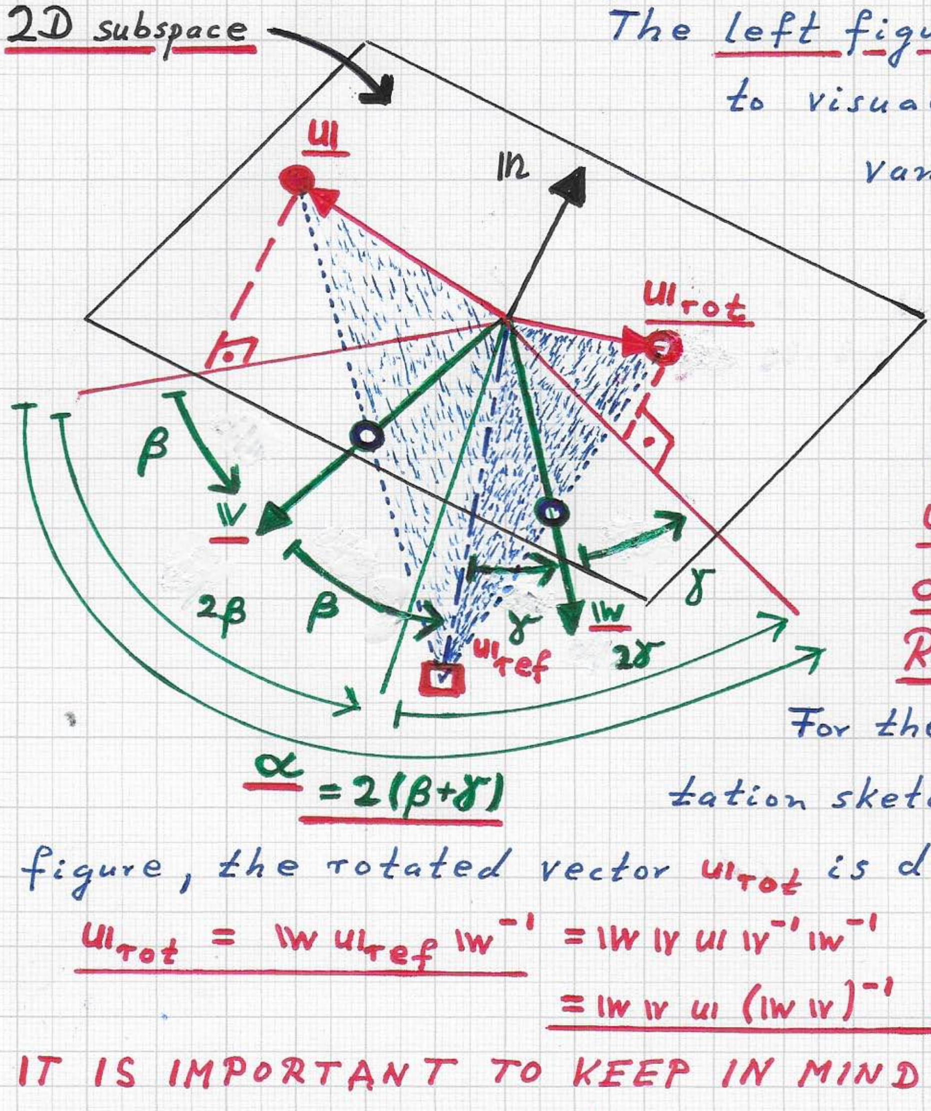
This representation of a rotator via trigonemetric functions points out the rotation in the plane, in 2D space, when done with complex numbers: $\mathbb{x}_{rot} = x_1' + i x_2' = (\cos\alpha + i \sin\alpha)(x_1 + i x_2)$, see page 23 (4-19-2024).



The left figure reminds us that a rotation in the plane by an angle α can be done by two reflections:
(i) use axis a_1 to map $\mathbb{x}$ to $\hat{\mathbb{x}}$;
(ii) use axis a_2 to map $\hat{\mathbb{x}}$ to $\mathbb{x}'$.
The result is $\mathbb{x}_{rot} = \mathbb{x}'$. ...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

- Laplacian eigenfunctions and neural networks:… It is challenging to sketch the general case of a rotation in a 2D subspace via two concatenated reflections in the geometric algebra setting.



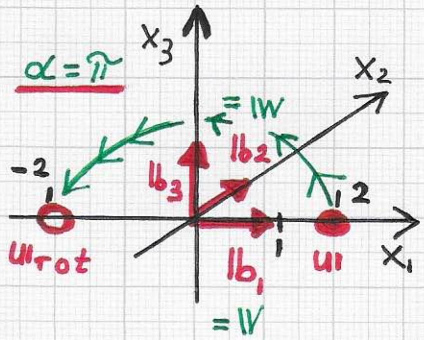
The left figure attempts to visualize the relevant "geometric primitives" and operations involved when rotating $\mathbb{u}$ by the angle α, using as ROTOR $R = \mathbb{w}\,\mathbb{v}$. For the general rotation sketched in this figure, the rotated vector $\mathbb{u}_{rot}$ is defined as

$$\mathbb{u}_{rot} = \mathbb{w}\,\mathbb{u}_{ref}\,\mathbb{w}^{-1} = \mathbb{w}\,\mathbb{v}\,\mathbb{u}\,\mathbb{v}^{-1}\mathbb{w}^{-1} = \mathbb{w}\,\mathbb{v}\,\mathbb{u}\,(\mathbb{w}\,\mathbb{v})^{-1} = R\,\mathbb{u}\,R^{-1}.$$

IT IS IMPORTANT TO KEEP IN MIND THAT TWO DIFFERENT ANGLES / ANGLE VALUES (β and γ in the above figure) DEFINE THE TOTAL ANGLE OF ROTATION: $\alpha = 2(\beta+\gamma)$. In the following, we provide simple examples to clarify the use of rotors and multiplication tables. …

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:...



The first example uses geometric algebra to perform a rotation in 3D space around the x_3-axis by π, see left figure. The vector $\mathbb{u} = (2,0,0)^T$ is mapped to $\mathbb{u}_{rot} = (-2,0,0)^T$ via the use of a **rotor**. The two vectors defining the rotor and the plane of rotation are $\mathbb{v} = (1,0,0)^T$ and $\mathbb{w} = (0,1,0)^T$. The orthonormal basis vectors are $\mathbb{b}_1 = (1,0,0)^T = \mathbb{v}$ and $\mathbb{b}_2 = (0,1,0)^T = \mathbb{w}$, with $\mathbb{b}_3 = (0,0,1)^T$ defining the axis of rotation. Thus, $\mathbb{v} = (v_1, v_2, v_3)^T = (1,0,0)^T$ and $\mathbb{w} = (w_1, w_2, w_3)^T = \ldots$ Further, the first reflection is performed relative to the x_1-axis, and the second reflection is done relative to the x_2-axis. **C**onsequently, for this scenario $\beta = 0$, $\gamma = \pi/2$ and $\alpha = 2(\beta + \gamma) = \pi$. We must compute the rotor R and R^{-1}:

$$R = \mathbb{w}\mathbb{v} = \langle \mathbb{w}, \mathbb{v} \rangle + (\mathbb{w} \wedge \mathbb{v}) = \langle (0,1,0)^T, (1,0,0)^T \rangle + ((0,1,0)^T \wedge (1,0,0)^T) =$$
$$= 0 + (w_1 v_2 - w_2 v_1)\,\mathbb{b}_{12} + (w_2 v_3 - w_3 v_2)\,\mathbb{b}_{23} + (w_3 v_1 - w_1 v_3)\,\mathbb{b}_{31} =$$
$$= 0 + (0-1)\,\mathbb{b}_{12} + (0-0)\,\mathbb{b}_{23} + (0-0)\,\mathbb{b}_{31} = 0 - 1\,\mathbb{b}_{12} =$$
$$= -\mathbb{b}_{12}\ ;$$

$$R^{-1} = (\mathbb{w}\mathbb{v})^{-1} = \mathbb{v}^{-1}\mathbb{w}^{-1} = \frac{\mathbb{v}}{\|\mathbb{v}\|^2}\frac{\mathbb{w}}{\|\mathbb{w}\|^2} = \frac{\mathbb{v}}{1}\frac{\mathbb{w}}{1} = \mathbb{v}\mathbb{w} = \langle \mathbb{v}, \mathbb{w} \rangle + (\mathbb{v} \wedge \mathbb{w}) =$$
$$= \langle \mathbb{w}, \mathbb{v} \rangle - (\mathbb{w} \wedge \mathbb{v}) = 0 + \mathbb{b}_{12} = \mathbb{b}_{12}\,.$$

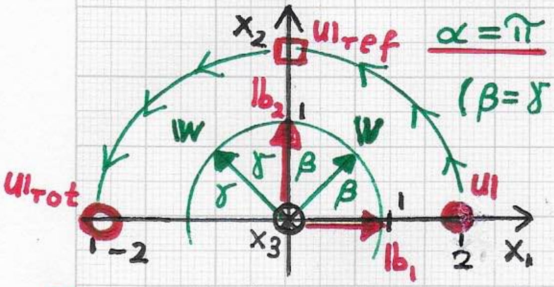
$$\Rightarrow \mathbb{u}_{rot} = R\,\mathbb{u}\,R^{-1} = -\mathbb{b}_{12}\,\mathbb{u}\,\mathbb{b}_{12} = -\mathbb{b}_{12}\,(2,0,0)^T\,\mathbb{b}_{12} = -\mathbb{b}_{12}\,2\,\mathbb{b}_1\,\mathbb{b}_{12} =$$
$$= 2(-\mathbb{b}_{12}\,\mathbb{b}_1\,\mathbb{b}_{12}) = 2(-\mathbb{b}_{12}\,\mathbb{b}_2) = -2\,\mathbb{b}_1 = (-2,0,0)^T.$$

...

Stratovan

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks: ...



Top-down view into (x_1, x_2)-plane, looking down the x_3-axis.

The second example considers a minor modification of the first example. The left figure shows the relevant data for this rotation in the (x_1, x_2)-plane: $\mathbf{u} = (2,0,0)^T$, $\mathbf{u}_{ref} = (0,2,0)^T$, $\mathbf{u}_{rot} = (-2,0,0)$; $\mathbf{b}_1 = (1,0,0)^T$, $\mathbf{b}_2 = (0,1,0)^T$, $\mathbf{b}_3 = (0,0,1)^T$; $\mathbf{v} = (\sqrt{2}/2, \sqrt{2}/2, 0)^T$, $\mathbf{w} = (-\sqrt{2}/2, \sqrt{2}/2, 0)^T$. For these specific values, one sees that $\mathbf{u}$ is mapped to $\mathbf{u}_{ref}$ and $\mathbf{u}_{ref}$ is mapped to $\mathbf{u}_{rot} = (\mathbf{u}_{ref})_{ref}$, where $\beta = \gamma$ and $\beta + \gamma = \pi/2$, defining the total rotation angle as $\alpha = \pi = 2(\beta + \gamma) = 2(\pi/4 + \pi/4)$. The computations are:

$$\mathbf{R} = \mathbf{w}\mathbf{v} = \langle \mathbf{w}, \mathbf{v} \rangle + (\mathbf{w} \wedge \mathbf{v}) = \langle (-\tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}}{2}, 0)^T, (\tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}}{2}, 0)^T \rangle +$$
$$+ \left((-\tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}}{2}, 0)^T \wedge (\tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}}{2}, 0)^T \right) = 0 + (w_1 v_2 - w_2 v_1)\mathbf{b}_{12} +$$
$$+ (w_2 v_3 - w_3 v_2)\mathbf{b}_{23} + (w_3 v_1 - w_1 v_3)\mathbf{b}_{31} = 0 - \mathbf{b}_{12} + 0\mathbf{b}_{23} + 0\mathbf{b}_{31} =$$
$$= 0 - \mathbf{b}_{12} \; ;$$

$$\mathbf{R}^{-1} = (\mathbf{w}\mathbf{v})^{-1} = \mathbf{v}^{-1}\mathbf{w}^{-1} = \frac{\mathbf{v}}{\|\mathbf{v}\|^2} \frac{\mathbf{w}}{\|\mathbf{w}\|^2} = \mathbf{v}\mathbf{w} = \ldots = \mathbf{b}_{12} .$$

$$\Rightarrow \mathbf{u}_{rot} = \mathbf{R}\mathbf{u}\mathbf{R}^{-1} = -\mathbf{b}_{12}\mathbf{u}\mathbf{b}_{12} = -\mathbf{b}_{12} 2\mathbf{b}_1 \mathbf{b}_{12} = (-2, 0, 0)^T .$$

• How can one use geometric algebra and rotors to accomplish a rotation in a 2D subspace in a general n-dimensional setting? ONE DEFINES TWO VECTORS, $\mathbf{v}$ AND $\mathbf{w}$, THAT DETERMINE THE ROTATION PLANE AND THE ROTATION ANGLE, $\alpha = 2\measuredangle(\mathbf{v}, \mathbf{w})$. ...