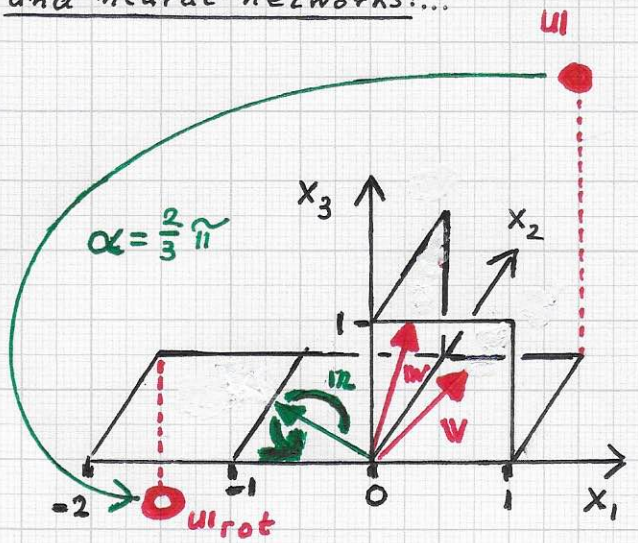
■ OBJECT AND MATERIAL EIGENFUNCTIONS - Cont'd.

• Laplacian eigenfunctions and neural networks:...



The third example considers a more complicated rotation in 3D space, see left figure. The point/positional vector to be rotated is $u1 = (1, 1, 2)^T$. Two vectors (normalized) define the plane in which u1 must be rotated; these two vectors are $iv = (\sqrt{2}/2, 0, \sqrt{2}/2)^T$ and $iw = (0, \sqrt{2}/2, \sqrt{2}/2)^T$. The angle between iv and iw is defined as $(\beta+\gamma)$: $\cos(\beta+\gamma) = \langle iv, iw \rangle / (\|iv\| \|iw\|) = 1/2 \Rightarrow \beta+\gamma = \pi/3$. Following the rules of geometric algebra, the final rotation angle to be used for the rotation is $\alpha = 2(\beta+\gamma) = 2/3\,\pi$. Further, the unit, oriented rotation axis **vector** is $in = (-\sqrt{3}/3, -\sqrt{3}/3, \sqrt{3}/3)^T$, defining the oriented normal vector of the rotation plane spanned by iv and iw, i.e., $in = (iv \times iw) / \|iv \times iw\|$. Before using geometric algebra to compute the coordinates of $u1_{rot}$, we employ standard linear algebra and apply a 3-by-3 rotation matrix to u1. This matrix is defined by the unit rotation axis vector in and the cos and sin values of the angle α.

See page 11 (12-2-2023) for the definition of the matrix.

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OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... For completeness, we include the rotation matrix here. Its components are defined by $\ln = (u, v, w)^T =$

$$\begin{bmatrix} u^2(1-c)+c & uv(1-c)-ws & uw(1-c)+vs \\ vu(1-c)+ws & v^2(1-c)+c & vw(1-c)-us \\ wu(1-c)-vs & wv(1-c)+us & w^2(1-c)+c \end{bmatrix}$$

$= (-\sqrt{3}/3, -\sqrt{3}/3, \sqrt{3}/3)^T$; $c = \cos\alpha = \cos(\frac{2}{3}\pi) = -1/2$; and $s = \sin\alpha = \sin(\frac{2}{3}\pi) = \sqrt{3}/2$. Thus, the matrix is

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}.$$

Consequently, we obtain the coordinates of u_{rot} as $(0\cdot1+0\cdot1-1\cdot2,\ 1\cdot1+0\cdot1+0\cdot2,\ 0\cdot1-1\cdot1+0\cdot2)^T = (-2, 1, -1)^T$.

We can now compute u_{rot} via geometric algebra, by using the geometric product with its associated multiplication table and the implied rotor R. The rotor defines u_{rot} as $u_{rot} = R u R^{-1}$, where $R = w\,v$ and R^{-1} is R's inverse. Therefore, we must compute the rotor and its inverse first:

$R = w\,v = (0, \sqrt{2}/2, \sqrt{2}/2)^T (\sqrt{2}/2, 0, \sqrt{2}/2)^T = \langle w, v\rangle + (w \wedge v) = \ldots$

{wedge product $(w_1, w_2, w_3)^T \wedge (v_1, v_2, v_3)^T =$
$= (w_1 v_2 - w_2 v_1)\,b_{12} + (w_2 v_3 - w_3 v_2)\,b_{23} + (w_3 v_1 - w_1 v_3)\,b_{31}$ }

$\ldots = \langle (0, \sqrt{2}/2, \sqrt{2}/2)^T, (\sqrt{2}/2, 0, \sqrt{2}/2)^T\rangle + (0, \sqrt{2}/2, \sqrt{2}/2)^T \wedge (\sqrt{2}/2, 0, \sqrt{2}/2)^T =$
$= 1/2 + (0 - 1/2)\,b_{12} + (1/2 - 0)\,b_{23} + (1/2 - 0)\,b_{31} =$
$= 1/2 + 1/2\,(-b_{12} + b_{23} + b_{31})$.

$R^{-1} = (w\,v)^{-1} = v^{-1} w^{-1} = v\,w = \langle v, w\rangle + (v \wedge w) =$
$= \langle w, v\rangle - (w \wedge v) = 1/2 - 1/2\,(-b_{12} + b_{23} + b_{31})$.

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■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... (• Note. One can compute RR^{-1} to see that their product is 1:

$$R = \tfrac{1}{2} + \tfrac{1}{2}(-lb_{12} + lb_{23} + lb_{31}) = \tfrac{1}{2} + \tfrac{1}{2}\mathbb{B}$$

$$R^{-1} = \tfrac{1}{2} - \tfrac{1}{2}(-lb_{12} + lb_{13} + lb_{31}) = \tfrac{1}{2} - \tfrac{1}{2}\mathbb{B}$$

$$\Rightarrow RR^{-1} = (\tfrac{1}{2} + \tfrac{1}{2}\mathbb{B})(\tfrac{1}{2} - \tfrac{1}{2}\mathbb{B}) =$$

$$= \tfrac{1}{4} - \tfrac{1}{4}\mathbb{B} + \tfrac{1}{4}\mathbb{B} - \tfrac{1}{4}\mathbb{B}^2 =$$

$$= \tfrac{1}{4} - \tfrac{1}{4}\mathbb{B}^2 = \tfrac{1}{4} - \tfrac{1}{4}(-lb_{12} + lb_{23} + lb_{31})^2 =$$

$$= \tfrac{1}{4} - \tfrac{1}{4}(-lb_{12} + lb_{23} + lb_{31})(-lb_{12} + lb_{23} + lb_{31}) =$$

$$= \tfrac{1}{4} - \tfrac{1}{4}(-1 + lb_{31} - lb_{23} - lb_{31} - 1 - lb_{12} + lb_{23} + lb_{12} - 1) =$$

$$= \tfrac{1}{4} - \tfrac{1}{4}(-3) = \tfrac{1}{4} + \tfrac{3}{4} = 1. \quad)$$

We can now compute u_{rot} as $u_{rot} = R\,u\,R^{-1}$:

$$u_{rot} = \left((\tfrac{1}{2} + \tfrac{1}{2}(-lb_{12} + lb_{23} + lb_{31}))(lb_1 + lb_2 + 2lb_3)\right)(\tfrac{1}{2} - \tfrac{1}{2}(-lb_{12} + lb_{23} + lb_{31})) =$$

$$= \left(\tfrac{1}{2}(lb_1 + lb_2 + 2lb_3) + \tfrac{1}{2}(-lb_{12} + lb_{23} + lb_{31})(lb_1 + lb_2 + 2lb_3)\right)$$
$$(\tfrac{1}{2} - \tfrac{1}{2}(-lb_{12} + lb_{23} + lb_{31})) =$$

$$= \left(\tfrac{1}{2}(lb_1 + lb_2 + 2lb_3) + \tfrac{1}{2}(-3lb_1 + 3lb_2)\right)\left(\tfrac{1}{2} - \tfrac{1}{2}(-lb_{12} + lb_{23} + lb_{31})\right) =$$

$$= \left(\tfrac{1}{2}(lb_1 + lb_2 + 2lb_3 - 3lb_1 + 3lb_2)\right)\left(\tfrac{1}{2} - \tfrac{1}{2}(-lb_{12} + lb_{23} + lb_{31})\right) =$$

$$= (-lb_1 + 2lb_2 + lb_3)\left(\tfrac{1}{2} - \tfrac{1}{2}(-lb_{12} + lb_{23} + lb_{31})\right) =$$

$$= \tfrac{1}{2}(-lb_1 + 2lb_2 + lb_3) - \tfrac{1}{2}(-lb_1 + 2lb_2 + lb_3)(-lb_{12} + lb_{23} + lb_{31}) =$$

$$= \tfrac{1}{2}(-lb_1 + 2lb_2 + lb_3 + (lb_1 - 2lb_2 - lb_3)(-lb_{12} + lb_{23} + lb_{31}) =$$

$$= \tfrac{1}{2}(-lb_1 + 2lb_2 + lb_3 + (-lb_2 + lb_{123} - lb_3$$
$$-2lb_1 - 2lb_3 - 2lb_{123} + lb_{123} + lb_2 - lb_1)) =$$

$$= \tfrac{1}{2}(-lb_1 + 2lb_2 + lb_3 - 3lb_1 - 3lb_3) = \tfrac{1}{2}(-4lb_1 + 2lb_2 - 2lb_3) =$$

$$= (-2lb_1 + lb_2 - lb_3) \triangleq (-2, 1, -1)^T.$$

In summary, the results obtained for the values of the coordinates of u_{rot} are once again −2, 1 and −1. ...

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■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:...

• Note. When using the "standard" linear algebra approach to rotate a point/positional vector around an oriented axis (passing through the origin), the 3-by-3 rotation matrix is defined by the three coordinate values of the normalized axis direction vector and the rotation angle value. When using the described geometric algebra approach to rotate a point/positional vector in an oriented plane (passing through the origin), the rotation plane is spanned by the normalized vectors $\mathbb{V}$ and $\mathbb{W}$ and the rotation angle value is twice the value of the angle between $\mathbb{V}$ and $\mathbb{W}$ (oriented "from $\mathbb{V}$ to $\mathbb{W}$"). Thus, the geometric algebra approach can be used in a "more straightforward manner" in the n-dimensional setting, since a rotation still must be performed in a 2D subspace (plane) in n-dimensional embedding space.

The fourth example uses the geometric set-up of the third example, but instead of rotating the specific point/positional vector $(1,1,2)^T$ we now perform the calculations for $\mathbb{u} = (u_1, u_2, u_3)^T$. One obtains:

$$\mathbb{u}_{rot} = \left(\left(\tfrac{1}{2} + \tfrac{1}{2}\left(-\mathbb{b}_{12} + \mathbb{b}_{23} + \mathbb{b}_{31}\right)\right)\left(u_1\mathbb{b}_1 + u_2\mathbb{b}_2 + u_3\mathbb{b}_3\right)\right)$$
$$\left(\tfrac{1}{2} - \tfrac{1}{2}\left(-\mathbb{b}_{12} + \mathbb{b}_{23} + \mathbb{b}_{31}\right)\right) =$$

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■ OBJECT AND MATERIAL EIGENFUNCTIONS - Cont'd.

• Laplacian eigenfunctions and neural networks:...

$$= \Big(\tfrac{1}{2}(u_1 lb_1 + u_2 lb_2 + u_3 lb_3) + \tfrac{1}{2}(-lb_{12} + lb_{23} + lb_{31})(u_1 lb_1 + u_2 lb_2 + u_3 lb_3) \Big) \Big(\tfrac{1}{2} - \tfrac{1}{2}(-lb_{12} + lb_{23} + lb_{31}) \Big) =$$

$$= \Big(\tfrac{1}{2}(u_1 lb_1 + u_2 lb_2 + u_3 lb_3) + \tfrac{1}{2}(u_1 lb_2 - u_2 lb_1 - u_3 lb_{123} + u_1 lb_{123} - u_2 lb_3 + u_3 lb_2 + u_1 lb_3 + u_2 lb_{123} - u_3 lb_1) \Big) \Big(\tfrac{1}{2} - \tfrac{1}{2}(-lb_{12} + lb_{23} + lb_{31}) \Big) =$$

$$= \Big(\tfrac{1}{2}(u_1 lb_1 + u_2 lb_2 + u_3 lb_3) + \tfrac{1}{2}((-u_2 - u_3) lb_1 + (u_1 + u_3) lb_2 + (u_1 - u_2) lb_3 + (u_1 + u_2 - u_3) lb_{123}) \Big) \Big(\tfrac{1}{2} - \tfrac{1}{2}(-lb_{12} + lb_{23} + lb_{31}) \Big) =$$

$$= \Big(\tfrac{1}{2}((u_1 - u_2 - u_3) lb_1 + (u_1 + u_2 + u_3) lb_2 + (u_1 - u_2 + u_3) lb_3 + (u_1 + u_2 - u_3) lb_{123}) \Big) \Big(\tfrac{1}{2} - \tfrac{1}{2}(-lb_{12} + lb_{23} + lb_{31}) \Big) =$$

$$= \tfrac{1}{4}\big((u_1 - u_2 - u_3) lb_1 + (u_1 + u_2 + u_3) lb_2 + (u_1 - u_2 + u_3) lb_3 + (u_1 + u_2 - u_3) lb_{123}\big)$$
$$- \tfrac{1}{4}\big((u_1 - u_2 - u_3) lb_1 + (u_1 + u_2 + u_3) lb_2 + (u_1 - u_2 + u_3) lb_3 + (u_1 + u_2 - u_3) lb_{123}\big)\big(-lb_{12} + lb_{23} + lb_{31}\big) =$$

$$= \tfrac{1}{4}\big((u_1 - u_2 - u_3) lb_1 + (u_1 + u_2 + u_3) lb_2 + (u_1 - u_2 + u_3) lb_3 + (u_1 + u_2 - u_3) lb_{123}\big)$$
$$- \tfrac{1}{4}\big((u_1 - u_2 - u_3)(-lb_2) + (u_1 - u_2 - u_3) lb_{123} + (u_1 - u_2 - u_3)(-lb_3)$$
$$+ (u_1 + u_2 + u_3) lb_1 + (u_1 + u_2 + u_3) lb_3 + (u_1 + u_2 + u_3) lb_{123}$$
$$+ (u_1 - u_2 + u_3)(-lb_{123}) + (u_1 - u_2 + u_3)(-lb_2) + (u_1 - u_2 + u_3) lb_1$$
$$+ (u_1 + u_2 - u_3) lb_3 + (u_1 + u_2 - u_3)(-lb_1) + (u_1 + u_2 - u_3)(-lb_2)\big) =$$

$$= \ldots = \tfrac{1}{4}\big((-4u_3) lb_1 + 4u_1 lb_2 + (-4u_2) lb_3\big) =$$

$$\cong -u_3 lb_1 + u_1 lb_2 - u_2 lb_3 \mathrel{\hat{=}} (-u_3, u_1, -u_2)^T.$$

This result agrees with the entries of the 3-by-3 rotation matrix on page 7 (4-30-2024): $lb_1 = (1,0,0)^T = 1lb_1 + 0lb_2 + 0lb_3 \mapsto (0,1,0)^T$; $lb_2 = (0,1,0)^T = 0lb_1 + 1lb_2 + 0lb_3 \mapsto (0,0,-1)^T$; $lb_3 = (0,0,1)^T = 0lb_1 + 0lb_2 + 1lb_3 \mapsto (-1,0,0)^T$. These vectors are the matrix columns.

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