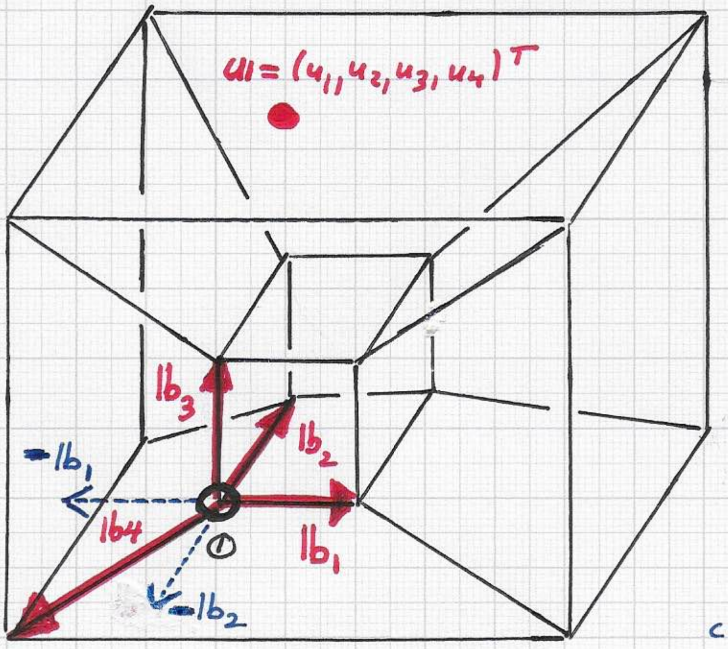
▪ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... An in-depth discussion of geometric algebra (GA) is not the goal of these high-level notes. GA can merely be viewed as a potential foundation for certain high-dimensional operations one must perform in the context of data processing and analysis for data classification purposes.



4D orthonormal basis: $\|\mathbb{b}_i\| = 1$; $i \neq j \Rightarrow \langle \mathbb{b}_i, \mathbb{b}_j \rangle = 0$.

Thus, we briefly consider a **4D GA-based** operation: a rotation in a plane by the angle π. The figure above is a sketch of a perspective projection of the unit 4D hyper-cube, showing origin ① and the four unit and mutually orthogonal basis vectors $\mathbb{b}_i$, $i = 1 \dots 4$. For example, one can use one of six possible "coordinate system planes" as a plane of rotation; the six possible planes are defined by the basis vector pairs $(\mathbb{b}_1, \mathbb{b}_2)$, $(\mathbb{b}_1, \mathbb{b}_3)$, $(\mathbb{b}_1, \mathbb{b}_4)$, $(\mathbb{b}_2, \mathbb{b}_3)$, $(\mathbb{b}_2, \mathbb{b}_4)$ and $(\mathbb{b}_3, \mathbb{b}_4)$. For simplicity, we select the 2D plane spanned by $\mathbb{b}_1$ and $\mathbb{b}_2$ to perform a rotation by π, using the GA-based approach.

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■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... We adopt the notation used in the four examples discussed on the previous pages to represent the rotation in the $(\mathbb{b}_1, \mathbb{b}_2)$-plane. We use $\mathbb{b}_1$ as $\mathbb{v}$ and $\mathbb{b}_2$ as $\mathbb{w}$, i.e., $\mathbb{v} = \mathbb{b}_1 = (1,0,0,0)^T$ and $\mathbb{w} = \mathbb{b}_2 = (0,1,0,0)^T$. The vectors $\mathbb{v}$ and $\mathbb{w}$ define the rotation plane and the value of the rotation angle: $\cos(\beta+\gamma) = \langle \mathbb{v}, \mathbb{w} \rangle / 1 = 0 \Rightarrow (\beta+\gamma) = \pi/2 \Rightarrow \alpha = 2(\beta+\gamma) = \pi$. The rotor R is

$$R = \mathbb{w}\mathbb{v} = (0,1,0,0)^T (1,0,0,0)^T = 0 + ((0,1,0,0)^T \wedge (1,0,0,0)^T) = 0 + (-\mathbb{b}_{12}) = -\mathbb{b}_{12}.$$

$$R^{-1} = (\mathbb{w}\mathbb{v})^{-1} = \mathbb{v}^{-1}\mathbb{w}^{-1} = \mathbb{v}\mathbb{w} = \ldots = 0 + ((1,0,0,0)^T \wedge (0,1,0,0)^T) = 0 + \mathbb{b}_{12} = \mathbb{b}_{12}.$$

$$\Rightarrow \mathbb{u}_{rot} = R\,\mathbb{u}\,R^{-1} = -\mathbb{b}_{12}\,\mathbb{u}\,\mathbb{b}_{12}.$$

An arbitrary point/positional vector $\mathbb{u}$ is represented as $\mathbb{u} = u_1 \mathbb{b}_1 + u_2 \mathbb{b}_2 + u_3 \mathbb{b}_3 + u_4 \mathbb{b}_4$. Therefore,

$$\begin{aligned}
\mathbb{u}_{rot} &= (-\mathbb{b}_{12}(u_1\mathbb{b}_1 + u_2\mathbb{b}_2 + u_3\mathbb{b}_3 + u_4\mathbb{b}_4))\,\mathbb{b}_{12} = \\
&= (-u_1\mathbb{b}_{12}\mathbb{b}_1 - u_2\mathbb{b}_{12}\mathbb{b}_2 - u_3\mathbb{b}_{12}\mathbb{b}_3 - u_4\mathbb{b}_{12}\mathbb{b}_4)\,\mathbb{b}_{12} = \\
&= (u_1\mathbb{b}_2 - u_2\mathbb{b}_1 + u_3\mathbb{b}_{321}^{*} - u_4\mathbb{b}_{412}^{*})\,\mathbb{b}_{12} = \\
&= -u_1\mathbb{b}_1 - u_2\mathbb{b}_2 + u_3\mathbb{b}_3 + u_4\mathbb{b}_4 \\
&\hat{=} (-u_1, -u_2, u_3, u_4)^T.
\end{aligned}$$

* multiplication tables

(Using standard matrix notation and algebra, this rotation is written as a product of matrix and vector:

$$\mathbb{u}_{rot} = \begin{bmatrix} c\alpha & -s\alpha & 0 & 0 \\ s\alpha & c\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} -u_1 \\ -u_2 \\ u_3 \\ u_4 \end{bmatrix}.)$$

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■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... •Note. The last example – rotation in a plane defining a 2D subspace in an embedding 4D space – was included to emphasize the point that **GA** can be considered as a foundation for operations to be done for classification purposes in high-dimensional space. ALGEBRA is a fundamental and "deep" branch in mathematics. Examples are linear algebra; geometric algebra (GA); projective geometric algebra (PGA); Clifford algebra; Grassmann(-Cayley) algebra; complex algebra; quaternion algebra; exterior algebra; conformal geometric algebra; hyper-complex algebra; and tensor algebra. In fact, some of these algebras are specialized/generalized versions of these algebras. One can also design, define and implement one's own algebra, based on exactly those objects and operations one is concerned with.

Multiplication tables are particularly important for the definition of a GA, for example. They define the results of multiplying a "basis element" (scalar, vector, bivector, trivector, quadrivector etc.) with another "basis element" in a specific GA, for n-dimensional space.

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■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... A specific aspect of the definition of a multiplication table is the signature of a GA. The signature is a triple of integers, (p, q, r), stating that p basis vectors square to 1, q basis vectors square to -1, and r basis vectors square to 0. For example there exist GAs $GA_{3,0,0}$ and $GA_{3,0,1}$ etc. Usually, multiplication tables list only the wedge products of GAs. A wedge product multiplication table is shown here (left). It shows the basis vector wedge products for basis vectors $\mathbb{b}_1$, $\mathbb{b}_2$, $\mathbb{b}_3$ ($GA_{3,0,0}$) and, via extension of the table, the basis vector wedge products for basis vectors $\mathbb{b}_1$, $\mathbb{b}_2$, $\mathbb{b}_3$ and $\mathbb{b}_4$ ($GA_{3,0,1}$). The size of these wedge product multiplication tables is defined by the dimension n. The left table shows the possible sets of scalar, vector, bivector, trivector, quadrivector, ..., n-vector. The number of possible k-vectors ($k = 0 \ldots n$) is 2^n. Thus, a multiplication table has 2^{2n} entries. ...

$\wedge$	1	$\mathbb{b}_1$	$\mathbb{b}_2$	$\mathbb{b}_3$	$\mathbb{b}_4$
1	1	$\mathbb{b}_1$	$\mathbb{b}_2$	$\mathbb{b}_3$	$\mathbb{b}_4$
$\mathbb{b}_1$	$\mathbb{b}_1$	1	$\mathbb{b}_{12}$	$-\mathbb{b}_{31}$	$-\mathbb{b}_{41}$
$\mathbb{b}_2$	$\mathbb{b}_2$	$-\mathbb{b}_{12}$	1	$\mathbb{b}_{23}$	$-\mathbb{b}_{42}$
$\mathbb{b}_3$	$\mathbb{b}_3$	$\mathbb{b}_{31}$	$-\mathbb{b}_{23}$	1	$-\mathbb{b}_{43}$
$\mathbb{b}_4$	$\mathbb{b}_4$	$\mathbb{b}_{41}$	$\mathbb{b}_{42}$	$\mathbb{b}_{43}$	0

n	...vectors, bivectors, ...	N
0	1	1
1	1 $\mathbb{b}_1$	2
2	1 $\mathbb{b}_1$ $\mathbb{b}_2$ $\mathbb{b}_{12}$	4
3	1 $\mathbb{b}_1$ $\mathbb{b}_2$ $\mathbb{b}_3$ $\mathbb{b}_{12}$ $\mathbb{b}_{13}$ $\mathbb{b}_{23}$ $\mathbb{b}_{123}$	8
4	1 $\mathbb{b}_1$ $\mathbb{b}_2$ $\mathbb{b}_3$ $\mathbb{b}_4$ $\mathbb{b}_{12}$ $\mathbb{b}_{13}$ $\mathbb{b}_{14}$ $\mathbb{b}_{23}$ $\mathbb{b}_{24}$ $\mathbb{b}_{34}$ $\mathbb{b}_{123}$ $\mathbb{b}_{124}$ $\mathbb{b}_{134}$ $\mathbb{b}_{234}$ $\mathbb{b}_{1234}$	16

▪ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... • **Note.** GA can be viewed as a potential "tool" for performing the operations that arise in high-dimensional data processing, especially geometry-based operations. Object and material recognition and classification involves high-dimensional feature space analysis, and data processing steps often are geometry-inspired. Therefore, the goal is use an existing or a "to-be-designed" algebra that supports robust, general, efficient and "understandable" geometry-based computations in high-dimensional space.

• **Note.** Considering our primary application – object and material classification – involving linear maps/transformations in n-dimensional space ($\mathbb{R}^n$), we are interested in potentially using a GA if one were to achieve more efficiency, "simplicity," generality etc. Thus, it is fundamental to be aware of the fact that every (non-zero) linear transformation f (on $\mathbb{R}^n$) extends to a linear transformation F (on GA^n), where $F(\mathbb{V} \wedge \mathbb{W}) = F(\mathbb{V}) \wedge F(\mathbb{W})$ holds for all multivectors $\mathbb{V}$ and $\mathbb{W}$. We are concerned with translation, scaling, projection (perpendicular), reflection and rotation in n-dimensional space. A GA provides an alternative for performing these transformations. •••