■ OBJECT AND MATERIAL EIGENFUNCTIONS - Cont'd.

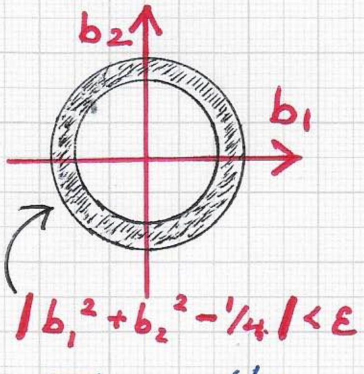
• Laplacian eigenfunctions and neural networks:... (A multitude of discrete, combinatorial optimization tools and software systems is available to perform high-dimensional design optimization for logic circuits. For example, simulated annealing, computational neural networks and genetic algorithms are used for this purpose. To achieve extremely high computational performance of a logic design, one can also use 2D or 3D field-programmable gate arrays (FPGAs).)

! **Dimensionality and variable reduction.** For computational and memory/storage efficiency it is imperative to reduce, for example, the chosen (or derived) variables from the initial variable set $b_1, \dots, b_n$ to the largest degree possible — still making sure that the required classification performance (degree of accuracy) is satisfied. The object/material samples used (for training) can lead to a substantial amount of potentially redundant, duplicative data; the number of variables used initially to characterize sample data ($b_1, \dots, b_n$) can be unnecessarily large, and there can exist relationships/dependencies between these variables.

...

■ OBJECT AND MATERIAL EIGENFUNCTIONS - Cont'd.

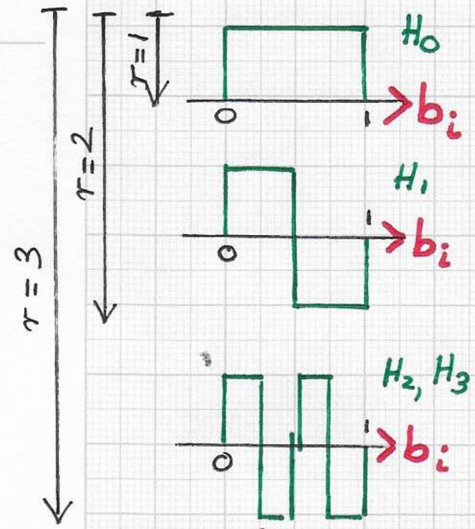
! • Laplacian eigenfunctions and neural networks:... **Inside/outside tests** can be used as an "alternative" to the use of multivariate class-identifying functions B_j, $j=1...N$. Of course, such tests are based on a test function, yielding as output/function value "in" or "out" or a real probability value between 0 and 1 indicating the probability for being inside. The terms inside/outside refer to the cases that a specific tuple $(b_1, ..., b_n)$ of a sample to be classified lies inside or outside the region in the used n-dimensional parameter space that indicates membership in class B_j. A "trivial example" is sketched in the left figure. Here, a sample to be classified belongs to class 3, for example, if the values of b_1 and b_2 satisfy $|b_1^2+b_2^2-1/4| < 1/100$.



Thus, the values of $b_3, ..., b_n$ are irrelevant or "void" to identify class-3 membership. Thus, determining the relevance of variables $b_1, b_2, ..., b_{n-1}, b_n$ for the class-membership tests for $B_1, B_2, ..., B_{N-1}, B_N$ is supremely important for making classification decisions in computationally minimal time.

▪ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

- Laplacian eigenfunctions and neural networks:... **Haar wavelets** are the most basic and simplest wavelets supporting a multiresolution representation of best-approximating expansions of functions — using the fully orthonormal basis of Haar basis functions and a simple underlying "dyadic grid" used to represent the $[0,1]^n$ domain via uniform, equal-volume hyper-cubes. (This is merely a high-level, simplistic description.)



One could consider the use of a Haar wavelet expansion for d-dimensional space:

$$H(b_1,\dots,b_d) = \sum_{i_d=0}^{2^{r-1}-1} \cdots \sum_{i_1=0}^{2^{r-1}-1} c_{i_1,\dots,i_d}\, H_{i_1}(b_1) \cdots H_{i_d}(b_d).$$

Such a tensor product expansion $H(b_1,\dots,b_d)$ would have to be computed as a best piecewise-constant approximation for a function B_j to perform optimally as a class-identifying function — for example representing class-membership probability. Here, d is the number of domain dimensions, and r is the dyadic resolution. The left figure and table sketch the Haar basis functions and list the number of coefficients, i.e., $2^{d(r-1)}$. ...

$r = 3 \Rightarrow 2^{3-1}-1 = 3 \Rightarrow H_0 \dots H_3$

d	r	$2^{d(r-1)}$
1	1, 2, 3, 4, ...	1, 2, 4, 8, ...
2	1, 2, 3, 4, ...	1, 4, 16, 64, ...
3	1, 2, 3, 4, ...	1, 8, 64, 512, ...
4	1, 2, 3, 4, ...	1, 16, 256, 4096, ...

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

- Laplacian eigenfunctions and neural networks:... It is possible to consider the following question: For a specific number of data samples (s), i.e., b-tuples with associated B-value, and a fixed number of dimensions (d) – or a fixed resolution (r) – what is the corresponding resolution value (r) – or the corresponding number of dimensions (d) – such that **the number of given data samples (s) equals the number of coefficients of a best approximating Haar wavelet expansion?** As shown in the table on the previous page, the total number of coefficients of an d-variate Haar wavelet expansion using r dyadic resolution levels is $2^{d(r-1)}$. Thus, for the case $s = 2^{d(r-1)}$ one obtains: $d(r-1) = \log_2(s)$. Therefore, the two crucial equations are: **(i) $r = 1 + \log_2(s)/d$; (ii) $d = \log_2(s)/(r-1)$.** For example, for $s = 2^{20} = 1048576$ data samples, $\log_2(2^{20}) = 20$; the resulting equations for r and d are $r = 1 + 20/d$ and $d = 20/(r-1)$. If one were to consider the use of $d=10$ dimensions, the total number of resolutions would be $r=3$; if one were to use a total number of resolutions $r=3$, the number of dimensions would be $d=10$.

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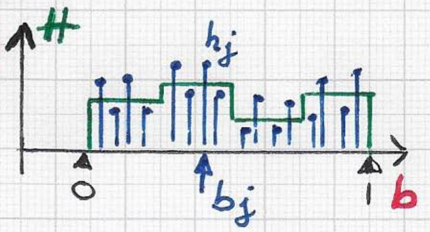
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... **Best Haar wavelet approximation.**



Considering the univariate case, a Haar wavelet expansion can be written as $\mathbf{H(b)} = \sum_{i=0}^{2^{r-1}-1} c_i H_i(b)$, where $b \in [0,1]$; $r \in \{1,2,3,\dots\}$ is the number of resolutions used; $\{H_0\}$, $\{H_1\}$, $\{H_2, H_3\}$, $\{H_4, H_5, H_6, H_7\}$, ... are the orthonormal basis function sets for scales $0, 1, 2, 3, \dots$; and c_i are real-valued coefficients. **The** top-left figure sketches an approximation $H(b)$, using four basis functions (H_0, H_1, H_2, H_3) to optimally approximate given data tuples (b_j, h_j), $j = 1 \dots s$. Thus, the constraints to be used are $H(b_j) = \sum_{i=0}^{2^{r-1}-1} c_i H_i(b_j) = h_j$, $j = 1 \dots s$. In matrix notation, this system is written as $\mathbb{H} \cdot \mathbb{c} = \mathbb{h}$:

$$\begin{bmatrix} H_0(b_1) & H_1(b_1) & \dots & H_{2^{r-1}-1}(b_1) \\ \vdots & \vdots & & \vdots \\ H_0(b_s) & H_1(b_s) & \dots & H_{2^{r-1}-1}(b_s) \end{bmatrix} \begin{bmatrix} c_0 \\ \vdots \\ c_{2^{r-1}-1} \end{bmatrix} = \begin{bmatrix} h_1 \\ \vdots \\ h_s \end{bmatrix}.$$

Here, $s \geq 2^{r-1}$.

The best approximation minimizing the least squares error is defined by $\mathbb{c} = (\mathbb{H}^T \mathbb{H})^{-1} \mathbb{H}\, \mathbb{h}$.

In the general setting, a high value of function $\mathcal{B}_j$ would indicate high probability for class-j membership. Concerning the original n-dimensional domain space, each function $\mathcal{B}_j$, $j = 1 \dots N$, would only be based on $d \ll n$ dimensions that permit the optimal definition of $\mathcal{B}_j$, via an orthonormal wavelet (Haar) expansion (largest coefficients).

...