## 4.2. Curvature approximation for triangulated two-dimensional surfaces

The graph of a bivariate function $f(x, y)$, $f$ in class $C^m$, $m \geq 2$, mapping an open set $U \subset I\!R^2$ into $I\!R$, can be interpreted as a regular parametric two-dimensional surface in three-dimensional space using the parametrization $x(u, v) = u$, $y(u, v) = v$, and $z(u, v) = f(u, v)$,

$$\mathbf{x}\,(\mathbf{u}) \;=\; \big(\; u, v, f(u, v)\;\big)^T, \qquad (u, v) \in D \;\subset\; I\!R^2. \tag{4.12.}$$

For this particular surface, one easily derives the formulae

$$\mathbf{x}_u \;=\; (1, 0, f_u)^T, \qquad \mathbf{x}_v \;=\; (0, 1, f_v)^T,$$

$$\mathbf{x}_{uu} \;=\; (0, 0, f_{uu})^T, \qquad \mathbf{x}_{uv} \;=\; (0, 0, f_{uv})^T, \qquad \mathbf{x}_{vv} \;=\; (0, 0, f_{vv})^T, \qquad \text{and}$$

$$\mathbf{n}\,(\mathbf{u}) \;=\; \frac{\mathbf{x}_u \times \mathbf{x}_v}{||\mathbf{x}_u \times \mathbf{x}_v||} \;=\; \frac{(-f_u, -f_v, 1)^T}{\sqrt{1 + {f_u}^2 + {f_v}^2}}. \tag{4.13.}$$

The first and second fundamental coefficients are therefore given by

$$E \;=\; 1 + {f_u}^2, \qquad F \;=\; f_u f_v \qquad G \;=\; 1 + {f_v}^2,$$

$$L \;=\; \frac{f_{uu}}{\sqrt{1 + {f_u}^2 + {f_v}^2}}, \qquad M \;=\; \frac{f_{uv}}{\sqrt{1 + {f_u}^2 + {f_v}^2}}, \qquad \text{and} \qquad N \;=\; \frac{f_{vv}}{\sqrt{1 + {f_u}^2 + {f_v}^2}}. \tag{4.14.}$$

The Gauss-Weingarten map for this particular surface is given by

$$-A \;=\; -\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \;=\; \frac{1}{l}\begin{pmatrix} f_{uu} & f_{uv} \\ f_{uv} & f_{vv} \end{pmatrix}\begin{pmatrix} 1 + {f_u}^2 & f_u f_v \\ f_u f_v & 1 + {f_v}^2 \end{pmatrix}^{-1}, \tag{4.15.}$$

where $l = \sqrt{1 + {f_u}^2 + {f_v}^2}$.

**Theorem 4.1.** *Each regular parametric two-dimensional surface* $\mathbf{x}(\mathbf{u})$ *of class* $m$, $m \geq 2$, *can locally be represented in the explicit form* $z = z(x, y)$ *which is at least* $C^2$. *Choosing a surface point* $\mathbf{x}_0$ *as origin of a local coordinate system and the* $z$-*axis in the same direction as the surface normal* $\mathbf{n}_0$ *at* $\mathbf{x}_0$ *(thus choosing the tangent plane at* $\mathbf{x}_0$ *as the* $xy$-*plane), the Taylor series for* $z$ *considering only the terms up to degree* 2 *is given by*

$$z\ (x, y) \ = \ \frac{1}{2}\ \left(c_{2,0}x^2 + 2c_{1,1}xy + c_{0,2}y^2\right), \tag{4.16.}$$

*choosing any* 2 *unit vectors in the* $xy$-*plane determining a right-handed orthonormal coordinate system. Rotating these* 2 *unit vectors appropriately yields the equation of the so-called* **osculating paraboloid** *at* $\mathbf{x}_0$,

$$z\ (x, y) \ = \ \frac{1}{2}\ \left(c^*_{2,0}x^2 + c^*_{0,2}y^2\right)$$

*such that the two principal curvatures at* $\mathbf{x}_0$ *coincide with the coefficients of this paraboloid,* $\kappa_1 = c^*_{2,0}$ *and* $\kappa_2 = c^*_{0,2}$.

**Proof.** See [Strubecker '58,'59] or [Struik '61].

The principal curvature approximation method to be introduced is based on bivariate polynomials. It is essential to prove a certain property of such functions before describing the approximation technique. Given an origin in the plane, the graph of a bivariate polynomial $f$ consisting of all the points in the set $\left\{ \left(\ x,\ y,\ f\ (x, y)\ \right)^T \mid x, y \in I\!R \right\}$ is independent of the choice of the orientation of the two unit vectors determining an orthonormal coordinate system for the plane. This fact implies that the principal curvatures of the graph, a two-dimensional surface, are independent of the two unit vectors as well.

**Lemma 4.1.** *The equation*

$$\sum_{k=0}^{i}(-1)^k\binom{i}{k}\ (x\cos^2\alpha+y\sin\alpha\cos\alpha)^{i-k}\ (-x\sin^2\alpha+y\sin\alpha\cos\alpha)^k=x^i \quad (4.17.)$$

*holds for all* $x,\ y,\ \alpha\in I\!R$ *and* $i\geq 0$.

**Proof.** It is easy to show that equation (4.17.) is valid for $i=0$:

$$1=x^0.$$

The induction hypothesis is made that equation (4.17.) is true for $i-1$. Thereby one proves that

$$\sum_{k=0}^{i}(-1)^k\binom{i}{k}\ (x\cos^2\alpha+y\sin\alpha\cos\alpha)^{i-k}\ (-x\sin^2\alpha+y\sin\alpha\cos\alpha)^k$$

$$=\big((x\cos^2\alpha+y\sin\alpha\cos\alpha)-(-x\sin^2\alpha+y\sin\alpha\cos\alpha)\big)$$

$$\sum_{k=0}^{i-1}(-1)^k\binom{i-1}{k}\ (x\cos^2\alpha+y\sin\alpha\cos\alpha)^{i-1-k}\ (-x\sin^2\alpha+y\sin\alpha\cos\alpha)^k$$

$$=x\ (\cos^2\alpha+\sin^2\alpha)\ x^{i-1}=x\ x^{i-1}=x^i.$$

q.e.d.

**Lemma 4.2.** *The equation*

$$\sum_{l=0}^{j}\binom{j}{l}\ (x\sin\alpha\cos\alpha+y\sin^2\alpha)^{j-l}\ (-x\sin\alpha\cos\alpha+y\cos^2\alpha)^l=y^j \qquad (4.18.)$$

*holds for all* $x,\ y,\ \alpha\in I\!R$ *and* $j\geq 0$.

**Proof.** Equation (4.18.) holds for $j = 0$:

$$1 = y^0.$$

Using the induction hypothesis that equation (4.18.) is true for $j - 1$, one proves that

$$\sum_{l=0}^{j} \binom{j}{l} \left(x \sin\alpha \cos\alpha + y \sin^2\alpha\right)^{j-l} \left(-x \sin\alpha \cos\alpha + y \cos^2\alpha\right)^{l}$$

$$= \big((x \sin\alpha \cos\alpha + y \sin^2\alpha) + (-x \sin\alpha \cos\alpha + y \cos^2\alpha)\big)$$

$$\sum_{l=0}^{j-1} \binom{j-1}{l} \left(x \sin\alpha \cos\alpha + y \sin^2\alpha\right)^{j-1-l} \left(-x \sin\alpha \cos\alpha + y \cos^2\alpha\right)^{l}$$

$$= y \ (\sin^2\alpha + \cos^2\alpha) \ y^{j-1} = y \ y^{j-1} = y^j.$$

q.e.d.

Lemma 4.1. and Lemma 4.2. are needed to prove the following theorem.

**Theorem 4.2.** *Let $f$ be the bivariate polynomial*

$$f\ (x, y) = \sum_{\substack{i+j \leq n \\ i,j \geq 0}} c_{i,j}\ x^i\ y^j, \tag{4.19.}$$

*where a point in the plane has coordinates $x$ and $y$ with respect to a coordinate system given by an origin $\mathbf{o}$ and two orthonormal basis vectors $\mathbf{d}_1$ and $\mathbf{d}_2$; rotating $\mathbf{d}_1$ and $\mathbf{d}_2$ around the origin $\mathbf{o}$ changes the representation of the bivariate polynomial, but not its graph.*

**Proof.** Let $\mathbf{d}_1$ and $\mathbf{d}_2$ be two unit vectors determining a first orthonormal coordinate system together with the origin $\mathbf{o}$, and let $\overline{\mathbf{d}_1}$ and $\overline{\mathbf{d}_2}$ be a second pair of unit vectors obtained by rotating $\mathbf{d}_1$ and $\mathbf{d}_2$ by an angle $\alpha$ around $\mathbf{o}$. A point in the plane may have coordinates $(x, y)^T$ with respect to the first coordinate system and coordinates

$$\begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{4.20.}$$

with respect to the second coordinate system. Assuming (4.19.) is the representation of the polynomial $f$ with respect to the first coordinate system, $f$ can be rewritten using the inverse map of (4.20.):

$$f\ (x = \overline{x}\cos\alpha - \overline{y}\sin\alpha,\ y = \overline{x}\sin\alpha + \overline{y}\cos\alpha)$$

$$= \sum_{\substack{i+j\le n \\ i,j\ge 0}} c_{i,j}\ (\overline{x}\cos\alpha - \overline{y}\sin\alpha)^i\ (\overline{x}\sin\alpha + \overline{y}\cos\alpha)^j. \tag{4.21.}$$

Evaluating $f$ at the point $(\overline{x}, \overline{y})^T = (x\cos\alpha + y\sin\alpha,\ -x\sin\alpha + y\cos\alpha)^T$, considering the binomial theorem, Lemma 4.1., and Lemma 4.2., one derives the equations

$$f\ (\overline{x} = x\cos\alpha + y\sin\alpha,\ \overline{y} = -x\sin\alpha + y\cos\alpha)$$

$$= \sum_{\substack{i+j\le n \\ i,j\ge 0}} c_{i,j}\ \big(\cos\alpha\ (x\cos\alpha + y\sin\alpha) - \sin\alpha\ (-x\sin\alpha + y\cos\alpha)\big)^i$$

$$\big(\sin\alpha\ (x\cos\alpha + y\sin\alpha) + \cos\alpha\ (-x\sin\alpha + y\cos\alpha)\big)^j$$

$$= \sum_{\substack{i+j\le n \\ i,j\ge 0}} c_{i,j}\ \Big( \sum_{k=0}^{i} (-1)^k \binom{i}{k} \big(\cos\alpha(x\cos\alpha + y\sin\alpha)\big)^{i-k} \big(\sin\alpha(-x\sin\alpha + y\cos\alpha)\big)^k$$

$$\sum_{l=0}^{j} \binom{j}{l}\ \big(\sin\alpha\ (x\cos\alpha + y\sin\alpha)\big)^{j-l}\ \big(\cos\alpha\ (-x\sin\alpha + y\cos\alpha)\big)^l\ \Big)$$

$$= \sum_{\substack{i+j\leq n\\ i,j\geq 0}} c_{i,j} \left( \sum_{k=0}^{i} (-1)^k \binom{i}{k} \left(x\cos^2\alpha + y\sin\alpha\cos\alpha\right)^{i-k} \left(-x\sin^2\alpha + y\sin\alpha\cos\alpha\right)^k \right.$$

$$\left. \sum_{l=0}^{j} \binom{j}{l} \left(x\sin\alpha\cos\alpha + y\sin^2\alpha\right)^{j-l} \left(-x\sin\alpha\cos\alpha + y\cos^2\alpha\right)^l \right)$$

$$= \sum_{\substack{i+j\leq n\\ i,j\geq 0}} c_{i,j}\; x^i\; y^j = f\;(x,y)$$

proving the theorem.

q.e.d.

The curvature approximation method is based on a localization of a two-dimensional triangulation. The local neighborhood around a point $\mathbf{x}_i$ is its platelet.

**Definition 4.9.** Given a two-dimensional triangulation in two- or three-dimensional space, the **platelet** $\mathcal{P}_i$ associated with a point $\mathbf{x}_i$ in the triangulation is the set of all triangles (determined by the index-triples $(j_1, j_2, j_3)$ specifying their vertices) sharing $\mathbf{x}_i$ as a common vertex,

$$\mathcal{P}_i \;=\; \bigcup \left\{\; (j_1, j_2, j_3) \mid i = j_1 \vee i = j_2 \vee i = j_3 \;\right\}. \tag{4.22.}$$

The vertices constituting $\mathcal{P}_i$ are referred to as **platelet points**.

In order to approximate the principal curvatures at a point $\mathbf{x}_i$ in a two-dimensional triangulation a bivariate polynomial is constructed for a certain neighborhood around this point. Considering the facts that a two-dimensional surface can locally be represented explicitly (Theorem 4.1.) and that the graph of a bivariate polynomial is independent of the orientation of the two unit vectors determining an orthonormal coordinate system for the plane (Theorem 4.2.), the following sequence of computations is proposed.

(i) Determine the platelet points associated with $\mathbf{x}_i$.

(ii) Compute the plane $P$ passing through $\mathbf{x}_i$ and having $\mathbf{n}_i$ (the normal at $\mathbf{x}_i$) as its normal.

(iii) Define an orthonormal coordinate system in $P$ with $\mathbf{x}_i$ as its origin and two arbitrary unit vectors in $P$.

(iv) Compute the distances of all platelet points from the plane $P$.

(v) Project all platelet points onto the plane, $P$ and represent their projections with respect to the local coordinate system in $P$.

(vi) Interpret the projections in $P$ as abscissae values and the distances of the original platelet points from $P$ as ordinate values.

(vii) Construct a bivariate polynomial $f$ approximating these ordinate values.

(viii) Compute the principal curvatures of $f$'s graph at $\mathbf{x}_i$.

All steps needing further explaining are discussed in greater detail. Let $\{\mathbf{y}_j = (x_j, y_j, z_j)^T \mid j = 0...n_i\}$ be the set of all platelet points associated with the point $\mathbf{x}_i$ such that $\mathbf{y}_0 = \mathbf{x}_i$, and let $\mathbf{n} = (n^x, n^y, n^z)^T$ be the outward unit normal vector at $\mathbf{y}_0$. The implicit equation for the plane $P$ is then given by

$$\begin{aligned}\mathbf{n} \cdot (\mathbf{x} - \mathbf{y}_0) &= n^x(x - x_0) + n^y(y - y_0) + n^z(z - z_0) \\ &= n^x x + n^y y + n^z z - (n^x x_0 + n^y y_0 + n^z z_0)\end{aligned}$$

$$= \; Ax + By + Cz + D \; = \; 0. \tag{4.23.}$$

Depending on the outward unit normal vector $\mathbf{n}$ one chooses a vector $\mathbf{a}$ perpendicular to $\mathbf{n}$ ($\mathbf{a} \cdot \mathbf{n} = 0$) among the possibilities

$$\mathbf{a} \;=\; \begin{cases} \frac{1}{n^x}\left(-(n^y + n^z), n^x, n^x\right)^T, & n^x \neq 0, \\ \frac{1}{n^y}\left(n^y, -(n^x + n^z), n^y\right)^T, & n^y \neq 0, \\ \frac{1}{n^z}\left(n^z, n^z, -(n^x + n^y)\right)^T, & n^z \neq 0, \end{cases}$$

in order to obtain the first unit basis vector $\mathbf{b}_1$,

$$\mathbf{b}_1 \;=\; \frac{\mathbf{a}}{||\mathbf{a}||}, \qquad ||\mathbf{a}|| \;=\; \sqrt{(\mathbf{a} \cdot \mathbf{a})}.$$

The second unit basis vector $\mathbf{b}_2$ is defined as the cross product of $\mathbf{n}$ and $\mathbf{b}_1$,

$$\mathbf{b}_2 \;=\; \mathbf{n} \times \mathbf{b}_1.$$

The perpendicular signed distances $d_j$, $j = 0 ... n_i$, of all platelet points $\mathbf{y}_j$ from the plane $P$ are

$$d_j \;=\; \mathrm{dist}\,(\mathbf{y}_j, P) \;=\; \frac{Ax_j + By_j + Cz_j + D}{\sqrt{A^2 + B^2 + C^2}} \;=\; Ax_j + By_j + Cz_j + D. \tag{4.24.}$$

Projecting all platelet points $\mathbf{y}_j$ onto $P$ yields the points $\mathbf{y}_j^P$,

$$\mathbf{y}_j^P \;=\; \mathbf{y}_j - d_j\;\mathbf{n}. \tag{4.25.}$$

Considering $\mathbf{y}_0$ as the origin and $\mathbf{b}_1$ and $\mathbf{b}_2$ as the two unit basis vectors of a local two-dimensional orthonormal coordinate system for the plane $P$, each point $\mathbf{y}_j^P$ in $P$ can be expressed in terms of that coordinate system. Therefore, one computes the difference vectors

$$\mathbf{d}_j = \mathbf{y}_j^P - \mathbf{y}_0, \qquad j = 0 ... n_i,$$

and expresses them as linear combinations of the two unit basis vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ in $P$. Each difference vector $\mathbf{d}_j$ can be represented in the form

$$\mathbf{d}_j \;=\; (\mathbf{d}_j \cdot \mathbf{b}_1)\ \mathbf{b}_1 + (\mathbf{d}_j \cdot \mathbf{b}_2)\ \mathbf{b}_2, \tag{4.26.}$$

defining the local coordinates $u_j$ and $v_j$ of the point $\mathbf{y}_j^P$ in terms of the local coordinate system:

$$\left(u_j, v_j\right)^T \;=\; \left(\mathbf{d}_j \cdot \mathbf{b}_1, \mathbf{d}_j \cdot \mathbf{b}_2\right)^T. \tag{4.27.}$$

Interpreting the local coordinates $u_j$ and $v_j$ as abscissae values and the signed distances $d_j$ as ordinate values (in direction of the normal $\mathbf{n}$), a polynomial $f(u, v)$ of degree two (see Theorem 4.1.) is constructed approximating these ordinate values. Forcing the polynomial $f$ to satisfy $f(0,0) = f_u(0,0) = f_v(0,0) = 0$, the constraints

$$f\ (u_j, v_j) \;=\; \frac{1}{2}\left(c_{2,0}{u_j}^2 + 2c_{1,1}u_jv_j + c_{0,2}{v_j}^2\right) \;=\; d_j, \qquad j = 1 \ldots n_i,$$

remain. Written in matrix representation these constraints are

$$\begin{pmatrix} {u_1}^2 & 2u_1v_1 & {v_1}^2 \\ \vdots & \vdots & \vdots \\ {u_{n_i}}^2 & 2u_{n_i}v_{n_i} & {v_{n_i}}^2 \end{pmatrix} \begin{pmatrix} c_{2,0} \\ c_{1,1} \\ c_{0,2} \end{pmatrix} \;=\; U\ \mathbf{c} \;=\; \mathbf{d} \;=\; \begin{pmatrix} d_1 \\ \vdots \\ d_{n_i} \end{pmatrix}. \tag{4.28.}$$

This overdetermined system of linear equations is solved using a least squares approach (see [Davis '75]). The resulting normal equations are

$$U^T\ U\ \mathbf{c} \;=\; U^T\ \mathbf{d}. \tag{4.29.}$$

Provided the determinant of $U^TU$ does not vanish this $3 \cdot 3-$system of linear equations can immediately be solved using Cramer's rule.

**Theorem 4.3.** *The principal curvatures $\kappa_1$ and $\kappa_2$ of the graph $\left( u, v, f(u,v) \right)^T \subset$ $I\!R^3$, $u, v \in I\!R$, of the bivariate polynomial*

$$f\ (u,v) \ = \ \frac{1}{2}\left(c_{2,0}u^2 + 2c_{1,1}uv + c_{0,2}v^2\right) \tag{4.30.}$$

*at the point $\left( 0, 0, f(0,0) \right)^T$ are given by the two real roots of the quadratic equation*

$$\kappa^2 - (c_{2,0} + c_{0,2})\ \kappa + c_{2,0}c_{0,2} - {c_{1,1}}^2 \ = \ 0. \tag{4.31.}$$

**Proof.** According to Definition 4.7. and equation (4.15.), the principal curvatures of $f$'s graph are the eigenvalues of the matrix

$$-A \ = \ \frac{1}{l}\begin{pmatrix} f_{uu} & f_{uv} \\ f_{uv} & f_{vv} \end{pmatrix}\begin{pmatrix} 1+{f_u}^2 & f_u f_v \\ f_u f_v & 1+{f_v}^2 \end{pmatrix}^{-1},$$

where $l = \sqrt{1 + {f_u}^2 + {f_v}^2}$. Evaluating $-A$ for $u = v = 0$, one obtains the matrix

$$-A \ = \ \begin{pmatrix} c_{2,0} & c_{1,1} \\ c_{1,1} & c_{0,2} \end{pmatrix},$$

having the characteristic polynomial in (4.31.).

q.e.d.

Solving the normal equations (4.29.) and determining the roots of the characteristic polynomial in (4.31.), one finally obtains the desired approximations for the principal curvatures at the point $\mathbf{x}_i$.

The above construction is illustrated in Figure 4.2. Shown are the platelet points around the point $\mathbf{x}_i$, the tangent plane $P$, its local orthonormal coordinate system (origin $\mathbf{x}_i$ and basis vectors $\mathbf{b}_1$ and $\mathbf{b}_2$), and the projections of the platelet points ($\mathbf{y}_j^P$) onto $P$.

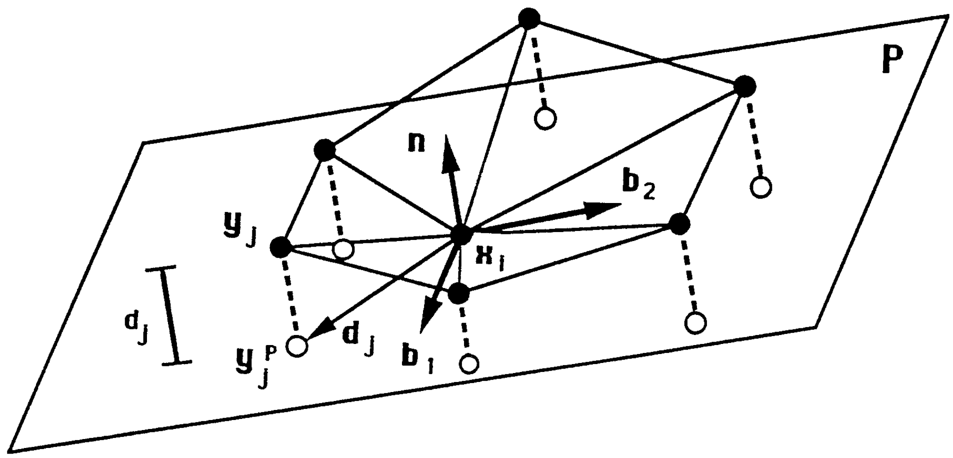


Fig. 4.2. Construction of a bivariate polynomial for platelet points in a two-dimensional triangulation.

The presented technique for principal curvature approximation is tested for graphs of several bivariate functions. The exact principal curvatures ${\kappa_1}^{ex}$ and ${\kappa_2}^{ex}$ are compared with the approximated principal curvatures ${\kappa_1}^{app}$ and ${\kappa_2}^{app}$; the exact mean curvature $H^{ex} = \frac{1}{2}\,({\kappa_1}^{ex} + {\kappa_2}^{ex})$ is compared with the average of the approximated principal curvatures $H^{app} = \frac{1}{2}\,({\kappa_1}^{app} + {\kappa_2}^{app})$ and the exact Gaussian curvature $K^{ex} = {\kappa_1}^{ex}{\kappa_2}^{ex}$ with the product of the approximated principal curvatures $K^{app} = {\kappa_1}^{app}{\kappa_2}^{app}$.

All bivariate test functions $f(x, y)$ are defined over $[-1, 1] \times [-1, 1]$ and evaluated on a $51 \cdot 51-$grid with equidistant spacing,

$$(x_i, y_j)^T \ = \ \Big( -1 + \frac{i}{25}, -1 + \frac{j}{25} \Big)^T, \qquad\qquad i, j = 0...50,$$

determining a finite set of three-dimensional points on their graphs,

$$\Big\{ \big( \ x_i, y_j, f(x_i, y_j) \ \big)^T \ | \ i, j = 0...50 \Big\}.$$

The triangulation of a function's graph is obtained by splitting each quadrilateral specified by its index quadruple

$$\big( \ (i, j), (i + 1, j), (i + 1, j + 1), (i, j + 1) \ \big)$$

into the two triangles $T^1_{i,j}$ and $T^2_{i,j}$ identified by their index triples,

$$T^1_{i,j} \ = \ \big( (i, j), (i{+}1, j), (i{+}1, j{+}1) \big) \quad \text{and} \quad T^2_{i,j} \ = \ \big( (i, j), (i{+}1, j{+}1), (i, j{+}1) \big).$$

The root-mean-square error (RMS error) is a common error measure and is computed for each test example and curvature type. The RMS error is defined as

$$\sqrt{\frac{1}{n} \sum_{i=0}^{n-1} ({f_i}^{ex} - {f_i}^{app})^2} \qquad (4.32.)$$

where $n$ is the total number of exact (or approximated) values ${f_i}^{ex}$ (${f_i}^{app}$). Here, $n$ equals $51 \cdot 51$, depending on the curvature type approximated ${f_i}^{ex}$ can represent the exact values for ${\kappa_1}^{ex}$, ${\kappa_2}^{ex}$, $H^{ex}$ or $K^{ex}$, and ${f_i}^{app}$ can represent the approximated values for ${\kappa_1}^{app}$, ${\kappa_2}^{app}$, $H^{app}$ or $K^{app}$, respectively. Table 4.1. summarizes the test results for the approximation of the principal curvatures, the mean and the Gaussian curvature.

Tab. 4.1. RMS errors of curvature approximation for graphs of bivariate functions.

| Function | $\kappa_1$ | $\kappa_2$ | $H$ | $K$ |
|---|---|---|---|---|
| 1. Plane: $.2\ (x+y)$. | 0 | 0 | 0 | 0 |
| 2. Cylinder: $\sqrt{2-x^2}$. | .000291 | .000035 | .000132 | .000025 |
| 3. Sphere: $\sqrt{4-(x^2+y^2)}$. | .000159 | .000046 | .000080 | .000080 |
| 4. Paraboloid: $.4\ (x^2+y^2)$. | .003073 | .001342 | .001358 | .001684 |
| 5. Hyperboloid: $.4\ (x^2-y^2)$. | .002058 | .002058 | .001057 | .001767 |
| 6. Monkey saddle: $.2\ (x^3-3xy^2)$. | .004483 | .004483 | .001591 | .007247 |
| 7. Cubic polynomial: $.15\ (x^3+2x^2y-xy+2y^2)$. | .002258 | .003598 | .001665 | .002242 |
| 8. Exponential function: $e^{-\frac{1}{2}(x^2+y^2)}$. | .001757 | .005546 | .002722 | .002602 |
| 9. Trigonometric function: $.1\ \big(\cos(\pi x)+\cos(\pi y)\big)$. | .002998 | .002821 | .001013 | .003541 |

In the following figures, the four particular curvatures used in Table 4.1. are mapped as textures onto the hyperboloid (function 5), the graph of the cubic polynomial (function 7) and the graph of the trigonometric function (function 9). Pairs of consecutive figures show the exact (upper figure) and the approximated curvatures (lower figure). The principal curvature $\kappa_1$ is visualized in the upper-left, $\kappa_2$ in the upper-right, the mean curvature $H$ in the lower-left and the Gaussian curvature $K$ in the lower-right corner of each figure. Figures 4.3. and 4.4. show the exact and approximated curvature values for function 5, Figures 4.5. and 4.6. for function 7, and Figures 4.7. and 4.8. for function 9.

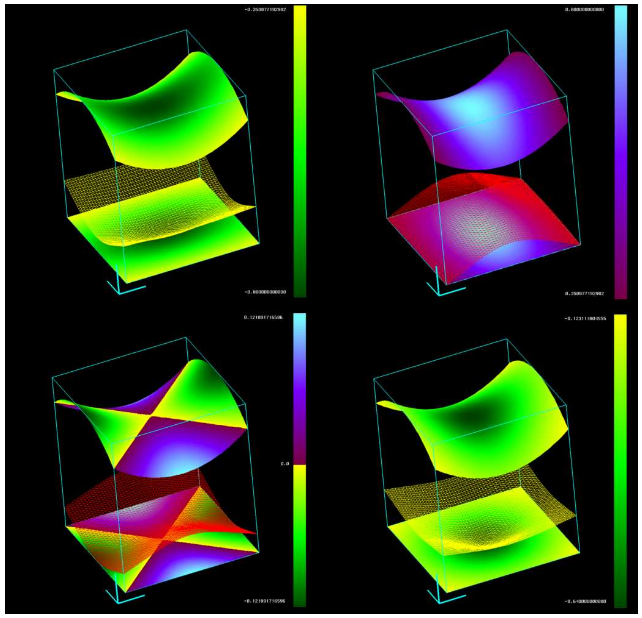

Fig. 4.3. Exact curvatures ${\kappa_1}^{ex}$, ${\kappa_2}^{ex}$, $H^{ex}$, and $K^{ex}$
on the graph of $f(x, y) = .4\ (x^2 - y^2),\ \ x, y \in [-1, 1]$.

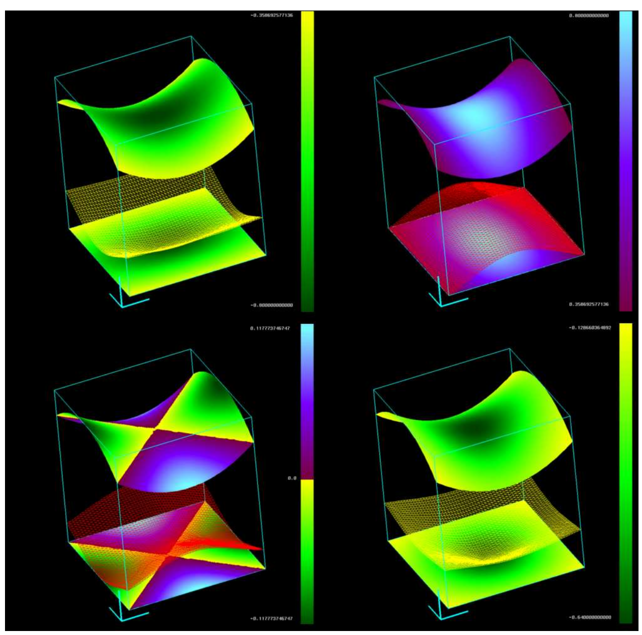

Fig. 4.4. Approximated curvatures ${\kappa_1}^{app}$, ${\kappa_2}^{app}$, $H^{app}$, and $K^{app}$
on the graph of $f(x, y) = .4\ (x^2 - y^2),\ \ x, y \in [-1, 1]$.

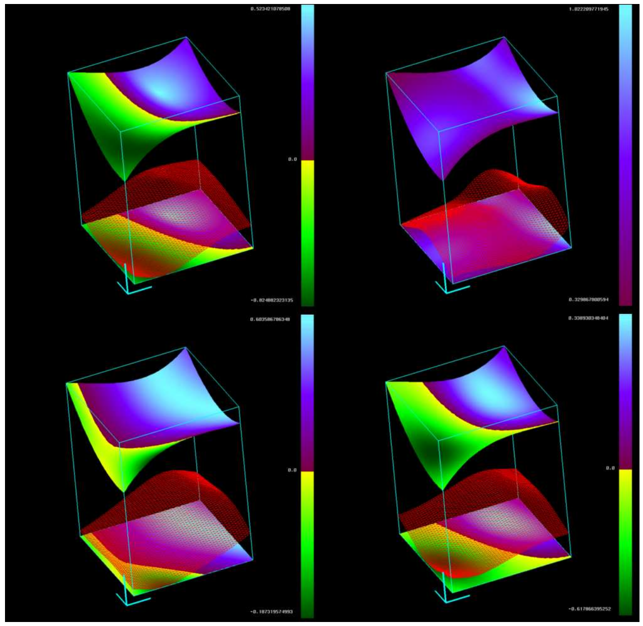

Fig. 4.5. Exact curvatures $\kappa_1{}^{ex}$, $\kappa_2{}^{ex}$, $H^{ex}$, and $K^{ex}$
on the graph of $f(x, y) = .15\ (x^3 + 2x^2y - xy + 2y^2),\ \ x, y \in [-1, 1]$.

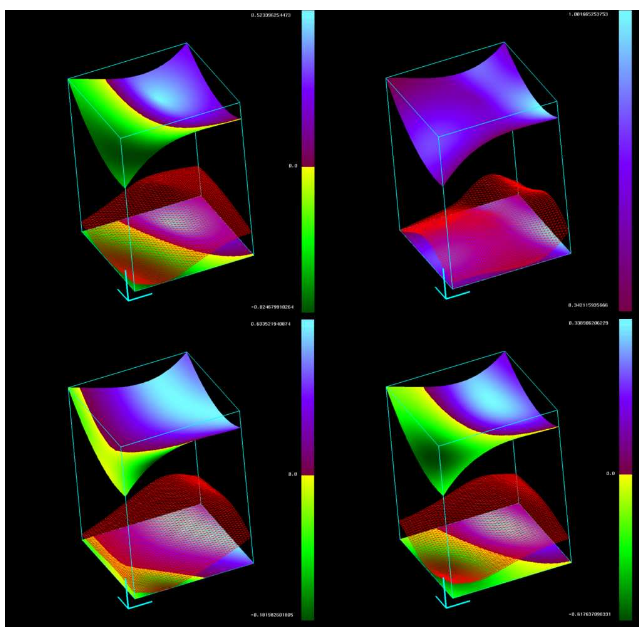

Fig. 4.6. Approximated curvatures $\kappa_1{}^{app}$, $\kappa_2{}^{app}$, $H^{app}$, and $K^{app}$
on the graph of $f(x, y) = .15\ (x^3 + 2x^2y - xy + 2y^2),\ \ x, y \in [-1, 1]$.

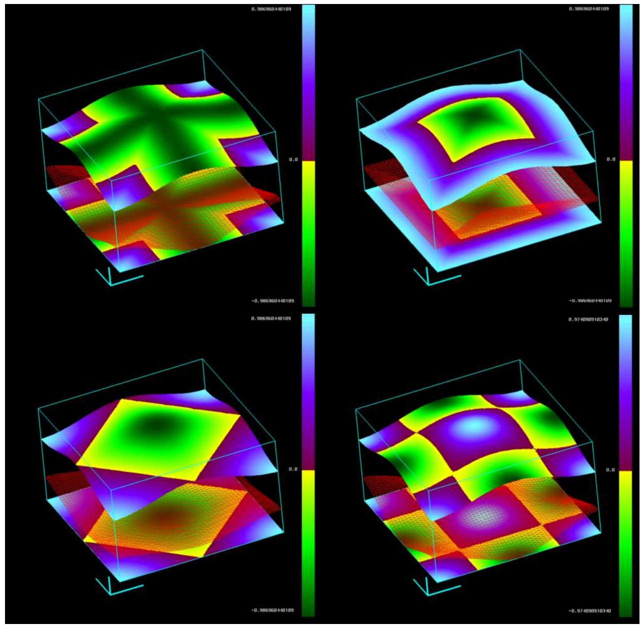

Fig. 4.7. Exact curvatures ${\kappa_1}^{ex}$, ${\kappa_2}^{ex}$, $H^{ex}$, and $K^{ex}$ on the graph of $f(x, y) = .1\ \big(\cos(\pi x) + \cos(\pi y)\big),\ x, y \in [-1, 1]$.

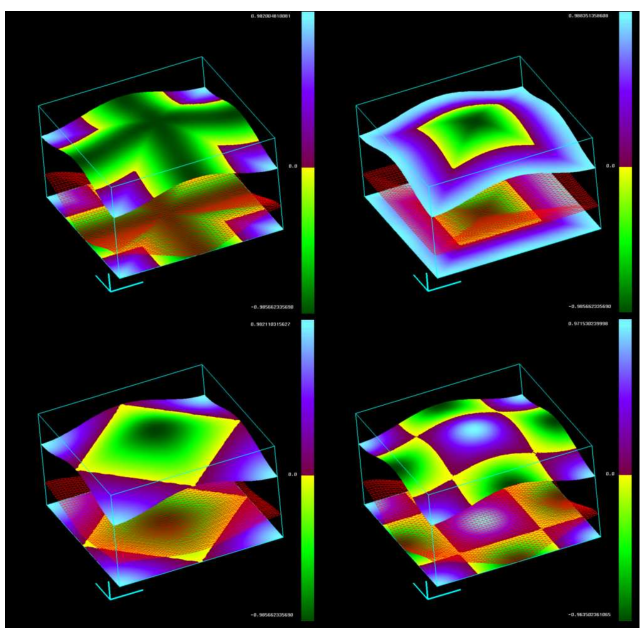

Fig. 4.8. Approximated curvatures ${\kappa_1}^{app}$, ${\kappa_2}^{app}$, $H^{app}$, and $K^{app}$ on the graph of $f(x, y) = .1\ \big(\cos(\pi x) + \cos(\pi y)\big),\ x, y \in [-1, 1]$.