5.2. Triangle reduction for triangulated two-dimensional surfaces

In order to determine the significance (weight) of a triangle in a two-dimensional triangulation, the principal curvatures at its vertices and its interior angles are considered. In this context, neither mean nor Gaussian curvature serve well as measures for a triangle's significance. The mean curvature at the point $(0,0,0)^T$ on the hyperboloid $\left(x, y, x^2 - y^2\right)^T$, $x, y \in I\!R$, is zero, and the Gaussian curvature at points in a plane and on a cylinder are both zero. Therefore, absolute curvature is used as an appropriate curvature measure.

Definition 5.10. The sum A of the absolute values of the principal curvatures κ_1 and κ_2 at a point $\mathbf{x}_0$ on the regular parametric surface $\mathbf{x}(\mathbf{u})$ is called the **absolute curvature**,

$$A \;=\; |\kappa_1| \;+\; |\kappa_2|. \tag{5.19.}$$

Since the overall goal is to establish an order in increasing significance on the finite set of triangles constituting a two-dimensional triangulation in three-dimensional space, each triangle is weighted by the three absolute curvatures at its vertices. The triangle with minimal absolute curvature is least significant, while the triangle with maximal absolute curvature is most significant. Later, the triangles are iteratively removed from the triangulation according to this order.

Lemma 5.1. *Denoting the interior angles of a triangle by* α_1, α_2, *and* α_3, *the range* *of the function*

$$f\ (\alpha_1, \alpha_2, \alpha_3) \ = \ \sum_{j=1}^{3} \cos \alpha_j \qquad (5.20.)$$

is the interval $[1, \frac{3}{2}]$.

Proof. Since $\sum_{j=1}^{3} \alpha_j = \pi$, it is sufficient to analyze the bivariate function

$$g\ (\alpha_1, \alpha_2) \ = \ \cos \alpha_1 + \cos \alpha_2 + \cos \big(\pi - (\alpha_1 + \alpha_2)\big)$$

on the domain $D = \big\{\ (\alpha_1, \alpha_2) \mid \alpha_1, \alpha_2 \geq 0,\ \ \alpha_1 + \alpha_2 \leq \pi\ \big\}$.

One easily proves that g equals one on D's boundary:

$$g\ (0, \alpha_2) \ = \ 1 + \cos \alpha_2 + \cos\ (\pi - \alpha_2) \ = \ 1,$$

$$g\ (\alpha_1, 0) \ = \ \cos \alpha_1 + 1 + \cos\ (\pi - \alpha_1) \ = \ 1, \qquad \text{and}$$

$$g\ (\alpha_1, \pi - \alpha_1) \ = \ \cos \alpha_1 + \cos\ (\pi - \alpha_1) + 1 \ = \ 1.$$

A critical point must satisfy the equations

$$\frac{\partial g}{\partial \alpha_1}\ (\alpha_1, \alpha_2) \ = \ - \sin \alpha_1 + \sin\big(\pi - (\alpha_1 + \alpha_2)\big) \ = \ 0 \qquad \text{and}$$

$$\frac{\partial g}{\partial \alpha_2}\ (\alpha_1, \alpha_2) \ = \ - \sin \alpha_2 + \sin\big(\pi - (\alpha_1 + \alpha_2)\big) \ = \ 0.$$

Therefore, $\sin \alpha_1 = \sin \alpha_2$ is a necessary condition which holds for $\alpha_1 = \alpha_2$ and $\alpha_2 = \pi - \alpha_1$.

The first case, $\alpha_1 = \alpha_2$, defines the univariate function

$$h\ (\alpha_1) \ = \ 2\ \ \cos \alpha_1 + \cos\ (\pi - 2\alpha_1)$$

having critical points at $\alpha_1 = 0$ and $\alpha_1 = \frac{\pi}{3}$, since $h'(0) = h'(\frac{\pi}{3}) = 0$. Considering only $\alpha_1 = \frac{\pi}{3}$ results in the function value $f(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}) = \frac{3}{2}$.

The second case, $\alpha_2 = \pi - \alpha_1$, defines part of D's boundary, where f equals one.

q.e.d.

Definition 5.11. The **angle weight** σ_i **of a triangle** T_i is given by

$$\sigma_i \;=\; \sigma\,(T_i) \;=\; 2\,\Big(\,\big(\sum_{j=1}^{3} \cos\alpha_j\,\big) - 1\,\Big) \quad \in\; [0,1], \tag{5.21.}$$

where α_j, $j = 1, 2, 3$, are T_i's interior angles.

Remark 5.3. The weight function in equation (5.21.) assigns maximum weight to an equilateral triangle and small weights to "long," "skinny" triangles.

Definition 5.12. The **curvature weight** ρ_i **of a triangle** T_i is given by the sum of the absolute curvatures at its vertices,

$$\rho_i \;=\; \rho\,(T_i) \;=\; \sum_{j=1}^{3} A_j, \tag{5.22.}$$

where A_j, $j = 1, 2, 3$, are the absolute curvatures at T_i's vertices.

Definition 5.13. The **weight** ω_i **of a triangle** T_i is given by

$$\omega_i \;=\; \omega\,(T_i) \;=\; \sigma_i\;\rho_i. \tag{5.23.}$$

The different steps concerning the removal of a single triangle T_i are discussed next. Assuming that the triangle platelet $\mathcal{TP}_i$ satisfies all the conditions stated in chapter 5.1., the triangle T_i is removed from the triangulation by replacing its vertices by one new point $\mathbf{p}$ whose construction must be described. This new point is connected to each point in $\mathcal{TP}_i$'s boundary vertex set, thus determining the new edges in the triangulation.

It is worth mentioning that this method of replacing a triangle does not affect the genus of the triangulation (precisely, the genus of the triangulated surface).

Definition 5.14. Given a two-dimensional triangulation $\mathcal{T}$, where each triangle has exactly three neighbors, the value

$$\chi \;=\; t - e + v, \tag{5.24.}$$

where t is the number of triangles, e the number of edges, and v the number of vertices in $\mathcal{T}$, is called the **Euler-Poincaré** characteristic of the implied C^0, piecewise linear surface. The **topological genus** is the value

$$1 - \frac{\chi}{2}. \tag{5.25.}$$

Remark 5.4. Considering a triangulation including triangles not having exactly three neighbors, equation (5.24.) must be modified. In this case, χ is defined as $t - e + v + 1$ ("open triangulation").

Theorem 5.4. *Replacing a triangle T_i whose corona $\mathcal{CP}_i$ is continuous and acyclic by a point* $\mathbf{p}$, *and constructing new edges by connecting the new point with each point in the boundary vertex set $\mathcal{B}_i^{ord}$, preserves the Euler-Poincaré characteristic.*

Proof. Replacing T_i by a point obviously reduces the number of vertices by two. Let k denote the number of edges (locally) being removed from the triangulation.

Then, the number of triangles is reduced by $(k-2)$. Considering these numbers and inserting them into equation (5.24.) yields

$$\chi \;=\; \big(t-(k-2)\big) - \big(e-k\big) + \big(v-2\big) \;=\; t-e+v,$$

proving that the Euler-Poincaré characteristic remains the same.

q.e.d.

In principle, there are two possibilities for the construction of the new point $\mathbf{p}$ replacing the triangle T_i. One possibility is to construct a bivariate function $f(u,v)$ using an appropriate coordinate system for the points determining T_i's triangle platelet and to evaluate f at $(0,0)^T$. The other possibility is to compute an implicit function $f(x,y,z)=0$, considering the same set of data points and to generate the new point by intersecting a line with the implicitly defined surface.

The first possibility is described next. The construction follows the same principle as the half-plane test (see chapter 5.1.), and the nomenclature from there is used. Depending on the corona $\mathcal{CP}_i^{ord}$, different choices for the origin $\mathbf{c}$ in the plane P are made.

- If the corona $\mathcal{CP}_i^{ord}$ is closed then the centroid of T_i is chosen.
- If T_i has three neighbors, but its corona is open, then the common vertex of T_i and the first and last triangle in $\mathcal{CP}_i^{ord}$ is chosen.
- If T_i has two neighbors, and the first and last triangle in $\mathcal{CP}_i^{ord}$ are both (are both not) neighbors of T_i then the mid-point of T_i's edge not shared by another triangle is chosen.

- If T_i has two neighbors, and the first (the last) triangle in $\mathcal{CP}_i^{ord}$ is a neighbor of T_i, and the last (the first) triangle in $\mathcal{CP}_i^{ord}$ is not a neighbor of T_i, then the vertex only shared by T_i and the first (the last) triangle in $\mathcal{CP}_i^{ord}$ is chosen.

- If T_i has a vertex not shared by another triangle then that vertex is chosen.

The different choices for the origin $\mathbf{c}$ are shown in Figure 5.4.

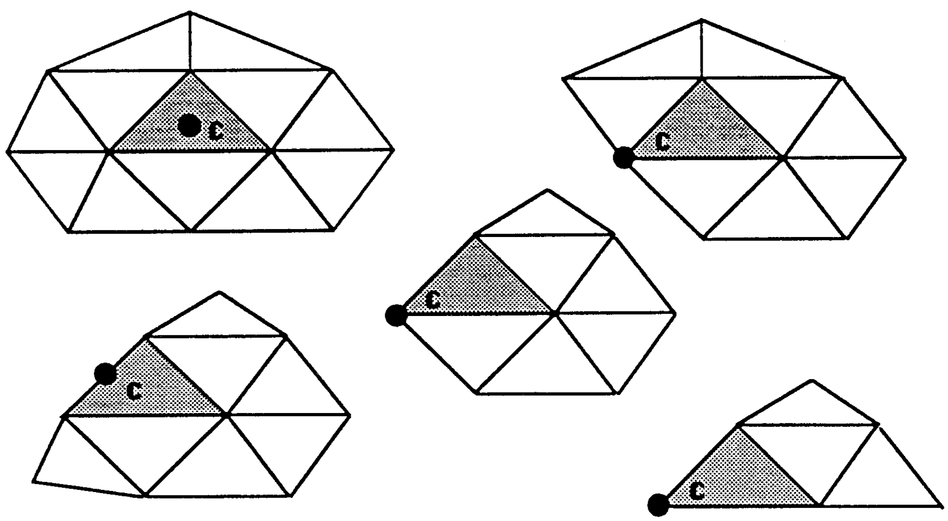

Fig. 5.4. Different choices for origin depending on triangle platelet.

Denoting the set of vertices in T_i's triangle platelet by $\{\mathbf{x}_1, ..., \mathbf{x}_{n_i}\}$, their associated two-dimensional coordinates in the plane P by $(u_j, v_j)^T$, and their distances from P by d_j, $j = 1...n_i$, again, a polynomial of degree two is constructed considering the constraints

$$f\ (u_j, v_j)\ =\ \sum_{\substack{i+k\le 2\\ i,k\ge 0}} c_{i,k}\ \ {u_j}^i\ {v_j}^k\ =\ d_j, \qquad j=1...n_i. \tag{5.26.}$$

In matrix representation these constraints are

$$\begin{pmatrix} {u_1}^2 & u_1v_1 & u_1 & {v_1}^2 & v_1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ {u_{n_i}}^2 & u_{n_i}v_{n_i} & u_{n_i} & {v_{n_i}}^2 & v_{n_i} & 1 \end{pmatrix} \begin{pmatrix} c_{2,0} \\ c_{1,1} \\ c_{1,0} \\ c_{0,2} \\ c_{0,1} \\ c_{0,0} \end{pmatrix} = U\ \mathbf{c}\ =\ \mathbf{d}\ = \begin{pmatrix} d_1 \\ \vdots \\ d_{n_i} \end{pmatrix}. \tag{5.27.}$$

Solving the normal equations

$$U^T\ U\ \mathbf{c}\ =\ U^T\ \mathbf{d} \tag{5.28.}$$

finally determines a local approximation in a function setting, provided that the determinant of U^TU does not vanish.

The new point $\mathbf{p}$ by which the triangle T_i is replaced is the point

$$\mathbf{p}\ =\ \mathbf{c} + f(0,0)\ \mathbf{n}, \tag{5.29.}$$

where $\mathbf{n}$ is T_i's outward unit normal vector (ordinate direction of f).

The second possibility to determine the new point $\mathbf{p}$ is the construction of a quadric $f(x,y,z)=0$ obtained by considering the constraints

$$f\ (x_j, y_j, z_j)\ =\ \sum_{\substack{i+k+l\le 2\\ i,k,l\ge 0}} c_{i,k,l}\ \ {x_j}^i\ {y_j}^k\ {z_j}^l\ =\ 0, \qquad j=1...n_i,$$

and the additional linear constraint

$$f\ (1,1,1)\ =\ \sum_{\substack{i+k+l\le 2\\ i,k,l\ge 0}} c_{i,k,l}\ =\ 1, \tag{5.30.}$$

where $(x_j, y_j, z_j)^T \in \{\mathbf{x}_1, ..., \mathbf{x}_{n_i}\}$ and $(1,1,1)^T \notin \{\mathbf{x}_1, ..., \mathbf{x}_{n_i}\}$, and $\{\mathbf{x}_1, ..., \mathbf{x}_{n_i}\}$ is the set of vertices in the triangle platelet $\mathcal{TP}_i$.

These conditions can be rewritten as

$$\begin{pmatrix} {x_1}^2 & x_1 y_1 & \dots & z_1 & 1 \\ {x_2}^2 & x_2 y_2 & \dots & z_2 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ {x_{n_i}}^2 & x_{n_i} y_{n_i} & \dots & z_{n_i} & 1 \\ 1 & 1 & \dots & 1 & 1 \end{pmatrix} \begin{pmatrix} c_{2,0,0} \\ c_{1,1,0} \\ \vdots \\ c_{0,0,1} \\ c_{0,0,0} \end{pmatrix} = X\ \mathbf{c} = \mathbf{y} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \qquad (5.31.)$$

Solving the normal equations

$$X^T\ X\ \mathbf{c} = X^T\ \mathbf{y} \qquad (5.32.)$$

finally determines a quadric surface locally approximating the vertices in the triangle platelet $\mathcal{TP}_i$.

Remark 5.5. The additional constraint $f(1,1,1) = 1$ is added, since the equation $f(x,y,z) = 0$ can be multiplied by any scalar still describing the same surface. Also, the condition $f(1,1,1) = 1$ defines an orientation of the quadric surface, its "outside" and "inside." However, this does not affect the absolute curvature at any point on the quadric.

Remark 5.6. If a triangle platelet does not provide enough vertices to uniquely determine the coefficients for a quadric (at least nine vertices) an appropriate subset of implicit surfaces is chosen, e.g.,

$$f\ (x,y,z) = \sum_{\substack{i+k+l \leq 2 \\ 0 \leq i,k,l \leq 1}} c_{i,k,l}\ \ x^i\ y^k\ z^l,$$

requiring less vertices.

The new point $\mathbf{p}$ is computed by intersecting the quadric surface with the line $\mathbf{x}(t) = \mathbf{c} + t\mathbf{n}$, $t \in I\!R$, where $\mathbf{c}$ is constructed as in the bivariate function setting, and $\mathbf{n}$ is T_i's outward unit normal vector. Inserting the linear expressions for the single components $x(t)$, $y(t)$, and $z(t)$ into the equation

$$f\,\left(x(t), y(t), z(t)\right) \;=\; \sum_{\substack{i+k+l\leq 2\\ i,k,l\geq 0}} c_{i,k,l}\;\left(x(t)\right)^i\,\left(y(t)\right)^k\,\left(z(t)\right)^l \;=\; 0$$

yields the quadratic equation

$$\begin{aligned}
t^2 \Big(\, & c_{2,0,0}\,(n^x)^2 + c_{1,1,0}\,n^x n^y + c_{1,0,1}\,n^x n^z + c_{0,2,0}\,(n^y)^2 + c_{0,1,1}\,n^y n^z + c_{0,0,2}\,(n^z)^2 \Big) \\
& + t\,\Big(\, 2\,(\,c_{2,0,0}\,c^x n^x + c_{0,2,0}\,c^y n^y + c_{0,0,2}\,c^z n^z\,) \\
& + c_{1,1,0}\,(c^x n^y + c^y n^x) + c_{1,0,1}\,(c^x n^z + c^z n^x) + c_{0,1,1}\,(c^y n^z + c^z n^y) \\
& + c_{1,0,0}\,n^x + c_{0,1,0}\,n^y + c_{0,0,1}\,n^z \Big) \\
& + \Big(\, c_{2,0,0}\,(c^x)^2 + c_{1,1,0}\,c^x c^y + c_{1,0,1}\,c^x c^z + c_{1,0,0}\,c^x + c_{0,2,0}\,(c^y)^2 \\
& + c_{0,1,1}\,c^y c^z + c_{0,1,0}\,c^y + c_{0,0,2}\,(c^z)^2 + c_{0,0,1}\,c^z + c_{0,0,0} \Big) \;=\; 0,
\end{aligned} \qquad (5.33.)$$

where $\mathbf{c} = (c^x, c^y, c^z)^T$ and $\mathbf{n} = (n^x, n^y, n^z)^T$.

Denoting the two (real) solutions of this equation by t_1 and t_2, the point in $\{\ \mathbf{c} + t_i\mathbf{n} \mid i = 1, 2\ \}$ having minimal distance to the plane P spanned by T_i is selected as the new point $\mathbf{p}$. Should the discriminant of the quadratic equation become negative, $\mathbf{c}$ is chosen as the point $\mathbf{p}$.

Having computed $\mathbf{p}$ by either the bivariate setting or by the implicit, trivariate approach, a first local re-triangulation of the boundary vertex set and $\mathbf{p}$ is obtained by connecting each vertex in $\mathcal{B}_i^{ord}$ with $\mathbf{p}$.

Figure 5.5. illustrates the removal of a triangle T_i with different triangle platelets and the re-triangulation of the remaining platelet boundary vertex set and the new point $\mathbf{p}$.

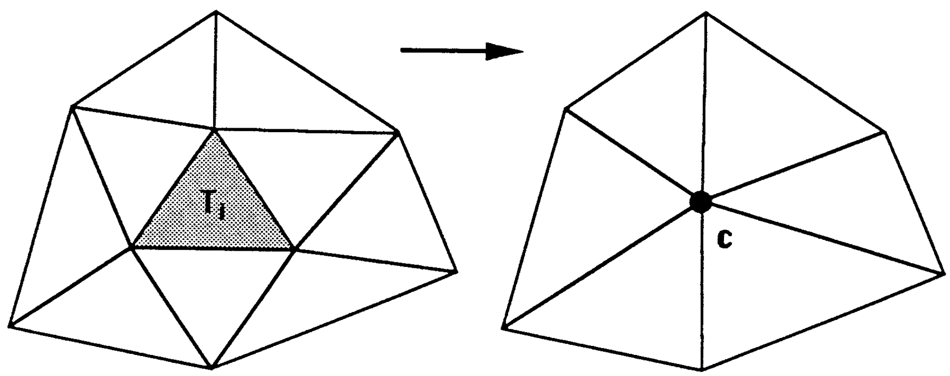

Fig. 5.5. Removal of triangle T_i and re-triangulation of boundary vertex set and new point.

Regardless of the method used for determining the new vertex $\mathbf{p}$, the absolute curvature must be computed for it, since the newly constructed triangles need to be weighted and appropriately inserted into the overall order of all triangles.

Assuming the new point $\mathbf{p}$ has been constructed by the bivariate function approach, the following theorem holds.

Theorem 5.5. *The principal curvatures* κ_1 *and* κ_2 *of the graph* $\left(u, v, f(u,v) \right)^T \subset \mathbb{R}^3$, $u, v \in \mathbb{R}$, *of the bivariate polynomial*

$$f\ (u,v) \ = \sum_{\substack{i+k\leq 2\\ i,k\geq 0}} c_{i,k}\ \ {u_j}^i\ {v_j}^k \tag{5.34.}$$

at the point $\left(0, 0, f(0,0) \right)^T$ *are given by the two real roots of the quadratic equation*

$$\det \begin{pmatrix} 2\ c_{2,0}\ (1 + {c_{0,1}}^2) - c_{1,1}c_{1,0}c_{0,1} - \kappa & -2\ c_{2,0}c_{1,0}c_{0,1} + c_{1,1}\ (1 + {c_{1,0}}^2) \\ c_{1,1}\ (1 + {c_{0,1}}^2) - 2\ c_{0,2}c_{1,0}c_{0,1} & -c_{1,1}c_{1,0}c_{0,1} + 2\ c_{0,2}\ (1 + {c_{1,0}}^2) - \kappa \end{pmatrix}$$

$$= \ 0. \tag{5.35.}$$

Proof. According to Definition 4.7. and equation (4.15.) the principal curvatures of f's graph are the eigenvalues of the matrix

$$-A \ = \ \frac{1}{l_1} \begin{pmatrix} f_{uu} & f_{uv} \\ f_{uv} & f_{vv} \end{pmatrix} \begin{pmatrix} 1 + {f_u}^2 & f_u f_v \\ f_u f_v & 1 + {f_v}^2 \end{pmatrix}^{-1},$$

where $l_1 = \sqrt{1 + {f_u}^2 + {f_v}^2}$. Evaluating $-A$ for $u = v = 0$, one obtains the matrix

$$\frac{1}{l_1} \begin{pmatrix} 2\ c_{2,0} & c_{1,1} \\ c_{1,1} & 2\ c_{0,2} \end{pmatrix} \begin{pmatrix} 1 + {c_{1,0}}^2 & c_{1,0}c_{0,1} \\ c_{1,0}c_{0,1} & 1 + {c_{0,1}}^2 \end{pmatrix}^{-1},$$

where $l_1 = \sqrt{1 + {c_{1,0}}^2 + {c_{0,1}}^2}$, having the characteristic equation (5.35.).

q.e.d.

The two roots of equation (5.35.), κ_1 and κ_2, determine the absolute curvature at the point $\mathbf{p}$ and therefore the curvature weights for all triangles sharing $\mathbf{p}$ as a common vertex.

Choosing the other possibility for computing $\mathbf{p}$, intersecting a quadric with a line, a formula is needed for calculating the principal curvatures for an arbitrary point on a quadric. It is well known in differential geometry how to compute the principal curvatures on an implicitly defined surface $f(x, y, z) = 0$.

Theorem 5.6. *Given the implicitly defined surface*

$$\mathcal{S} \ = \ \left\{ \ \mathbf{x} \in I\!R^3 \ \middle| \ f(\mathbf{x}) = 0 \ \right\},$$

where f is some polynomial with a non-vanishing gradient $\bigtriangledown f$ for all points in $\mathcal{S}$, and defining the surface outward normal vector at the surface point $\mathbf{x}_0 = (x_0, y_0, z_0)^T \in \mathcal{S}$ as

$$\mathbf{n} \;=\; \frac{1}{2}\,\left(\,\bigtriangledown f\,\left(x_0, y_0, z_0\right)\,\right)^T$$

the tangent space at $\mathbf{x}_0$ is spanned by the two vectors $\mathbf{b}_1$ and $\mathbf{b}_2$, where $\mathbf{b}_1$ is any unit vector perpendicular to $\mathbf{n}$ and $\mathbf{b}_2$ is the normalized cross product of $\mathbf{n}$ and $\mathbf{b}_1$,

$$\mathbf{b}_1 \cdot \mathbf{n} \;=\; 0, \quad \sqrt{\mathbf{b}_1 \cdot \mathbf{b}_1} \;=\; 1, \qquad \textit{and} \qquad \mathbf{b}_2 \;=\; \frac{\mathbf{n} \times \mathbf{b}_1}{||\mathbf{n} \times \mathbf{b}_1||}. \tag{5.36.}$$

Introducing the vector valued differential operator $\mathbf{D_d}$ as

$$\mathbf{D_d}\; f\,\Bigg|_{\mathbf{x}_0} \;=\; \frac{1}{2}\begin{pmatrix} f_{xx}d^x + f_{xy}d^y + f_{xz}d^z \\ f_{xy}d^x + f_{yy}d^y + f_{yz}d^z \\ f_{xz}d^x + f_{yz}d^y + f_{zz}d^z \end{pmatrix}\Bigg|_{\mathbf{x}_0}, \tag{5.37.}$$

where $\mathbf{d} = (d^x, d^y, d^z)^T$ is a directional vector and $f_{xx}, ..., f_{zz}$ denote the second order partial derivatives of f, the mean and Gaussian curvature at $\mathbf{x}_0$ are

$$H \;=\; -\,\frac{\mathbf{n}\;\cdot\;\left(\mathbf{D}_{\mathbf{b}_1}f \times \mathbf{b}_2 \;+\; \mathbf{b}_1 \times \mathbf{D}_{\mathbf{b}_2}f\right)}{2\;||\;\mathbf{n}\;||^{\;3}}\Bigg|_{\mathbf{x}_0} \qquad \textit{and}$$

$$K \;=\; \frac{\mathbf{n}\;\cdot\;\left(\mathbf{D}_{\mathbf{b}_1}f \times \mathbf{D}_{\mathbf{b}_2}f\right)}{||\;\mathbf{n}\;||^{\;4}}\Bigg|_{\mathbf{x}_0}, \tag{5.38.}$$

where $||\mathbf{n}|| = \sqrt{\mathbf{n}\cdot\mathbf{n}}$. The principal curvatures, κ_1 and κ_2, are related to the mean and Gaussian curvature by

$$\kappa_{1/2} \;=\; H \;\pm\; \sqrt{H^2 - K}. \tag{5.39.}$$

Proof. See [O'Neill '69], chapter 5.

These formulae can easily be applied to a quadric, thus determining the absolute curvature at the new point $\mathbf{p}$.

As already mentioned above, a natural way to triangulate the platelet boundary vertex set $\mathcal{B}_i^{ord}$ and the additional, new point $\mathbf{p}$ is to construct edges from $\mathbf{p}$ to each vertex in $\mathcal{B}_i^{ord}$, thus defining the (local) re-triangulation $\mathcal{N}_i$,

$$\mathcal{N}_i \;=\; \bigcup \left\{\; T_j \;=\; (l_0, l_k, l_{(k+1)\mathrm{mod}(N_i+1)}) \mid k = 3...m \;\right\}, \tag{5.40.}$$

where m equals either $N_i - 1$ (open corona) or N_i (closed corona), and $\{\mathbf{x}_{l_0}, ..., \mathbf{x}_{l_{N_i}}\}$ is the set of vertices in $\mathcal{TP}_i$ (see Definition 5.6.). This means that the new point $\mathbf{p}$ "inherits" the index of the first vertex, l_0, of the removed triangle, T_i, and vertices with indices l_1 and l_2 no longer exist.

In order to obtain a (local) re-triangulation consisting of triangles with high angle weights, an iterative, Lawson-like algorithm is applied to the set of newly constructed triangles in the set $\mathcal{N}_i$, therefore iteratively modifying the triangulation $\mathcal{N}_i$ (see [Lawson '77]). The idea is to swap diagonals of quadrilaterals, edges shared by two neighbors in $\mathcal{N}_i$. Nevertheless, diagonals are swapped only if the region obtained by projecting a quadrilateral onto the plane P determined by the removed triangle T_i is convex.

Definition 5.15. The **quadrilateral** formed by the line segments $\overline{\mathbf{ab}}$, $\overline{\mathbf{bc}}$, $\overline{\mathbf{cd}}$, $\overline{\mathbf{da}}$, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d} \in I\!R^3$, **has a convex projection with respect to a plane** P if the quadrilateral in P, formed by the line segments $\overline{\mathbf{a}^P\mathbf{b}^P}$, $\overline{\mathbf{b}^P\mathbf{c}^P}$, $\overline{\mathbf{c}^P\mathbf{d}^P}$, $\overline{\mathbf{d}^P\mathbf{a}^P}$, where $\mathbf{a}^P$, $\mathbf{b}^P$, $\mathbf{c}^P$, and $\mathbf{d}^P$ are the orthogonal projections of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ onto P, describes the polygonal boundary of a convex region in P.

Quadrilaterals formed by neighbor triangles in $\mathcal{N}_i$ satisfying this condition are swapped as long as this results in an increase of the minimum of the angle weights in the set of new triangles. This strategy finally terminates, since there is only a limited number of possible triangulations and one of them maximizes the minimum angle weight in $\mathcal{N}_i$.

Figure 5.6. shows the effect of swapping diagonals in a local re-triangulation, demonstrating the improvement of angle weights.

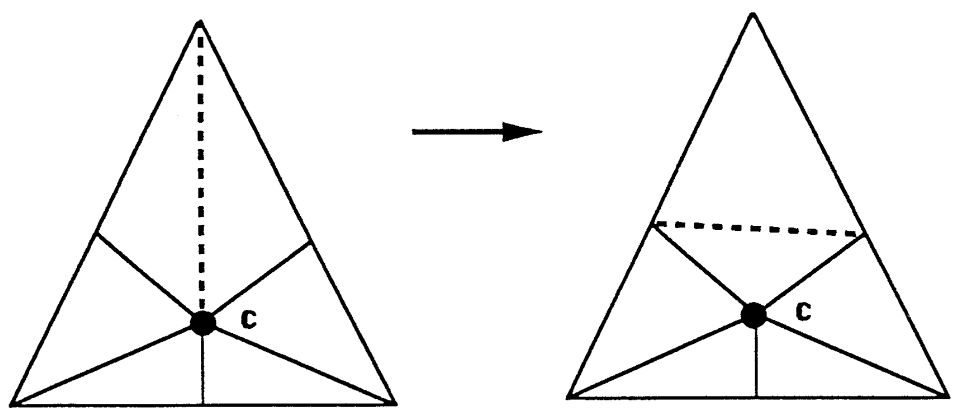

Fig. 5.6. Increasing angle weights of triangles in local re-triangulation (original and improved re-triangulation).

Having computed the weights of all triangles in the set $\mathcal{N}_i$ they are inserted into the overall order of all triangles. Simplified, the triangle reduction algorithm can be summarized as follows.

Algorithm 5.1. *Triangle reduction by iterative triangle removal*

Input: table $\mathcal{T}$ of N_0 triangles (including neighborhood information), table $\mathcal{V}$ of vertices (including principal curvatures), and a percentage $p \in [0, 100]$.
Output: reduced table $\hat{\mathcal{T}}$ of triangles and reduced table $\hat{\mathcal{V}}$ of vertices.

compute weights for each triangle in $\mathcal{T}$;
while number of triangles is greater than $\frac{p}{100}\ N_0$
(
among all triangles having a continuous, acyclic corona and passing the half-plane test determine the triangle T_i with minimal weight ω_{min};
remove triangle T_i from triangulation (using either a bivariate or a trivariate, implicit least squares approximation);
compute a first (local) re-triangulation;
compute the curvature weights for all new triangles;
improve the (local) re-triangulation by maximizing the minimum angle weight;
compute weights for new triangles;
)

Remark 5.7. If two triangles T_i and T_j exist both having minimum weight ω_{min} any of the two can be removed first. Removing either one of them first does not affect the final result as long as the triangle platelets $\mathcal{TP}_i$ and $\mathcal{TP}_j$ have an empty intersection.

Remark 5.8. Considering triangles not surrounded by a closed corona as boundary triangles of a triangulation, it is possible to force the algorithm not to remove such triangles. This, however, leads to reduced triangulations keeping a high density of vertices on the boundary. This problem, of course, does not arise in the case of reducing triangulations of closed surfaces.

Remark 5.9. In each iteration step triangles obtained by the local re-triangulation procedure are marked as "new" triangles which can not be removed at once in the next iteration step. Only if no triangle among the “old” ones can be removed, all new triangles can be considered for removal.

Remark 5.10. The termination criterion in the above algorithm (using a certain percentage of the original number of triangles) can be modified in the case of a triangulation obtained from the graph of a bivariate function. Then, the RMS error can be computed interpreting original and an intermediate triangulation as piecewise linear functions. As soon as the RMS error exceeds a certain tolerance ϵ, the algorithm stops.

Remark 5.11. It is possible that the reduction algorithm can not find any triangle having a continuous, acyclic corona and passing the half-plane test.

In the following examples, a triangle is always replaced by a point using the bivariate function approach for computing a new point $\mathbf{p}$. Each of the next three figures shows an original (upper-left) and three reduced triangulations for a reduction in the number of triangles by 50% (upper-right), 80% (lower-left), and 90% (lower-right). The initial triangulations for Figures 5.7., 5.8., and 5.9. are obtained by evaluating particular bivariate functions on $[-1,1] \times [-1,1]$ using a domain grid with equidistant spacing determining points on their graphs.

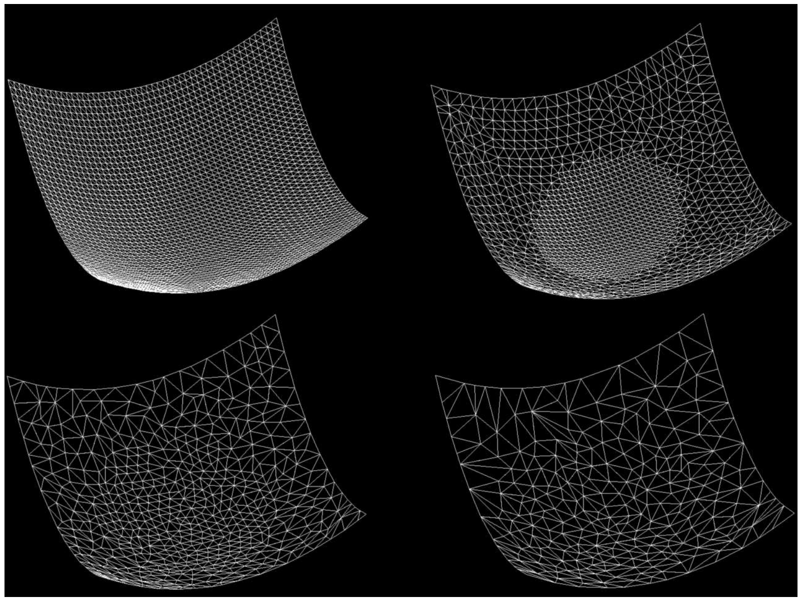

Fig. 5.7. Triangle reduction of 50%, 80%, and 90%
for the graph of $f(x,y) = .4\ (x^2 + y^2),\ x, y \in [-1, 1]$.

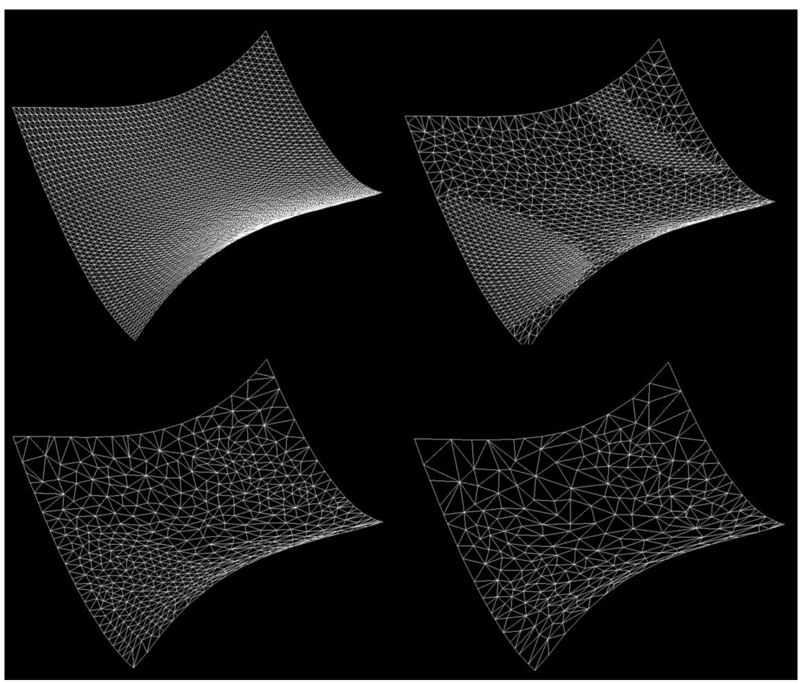

Fig. 5.8. Triangle reduction of 50%, 80%, and 90%
for the graph of $f(x,y) = .15\ (x^3 + 2x^2y - xy + 2y^2),\ x, y \in [-1, 1]$.

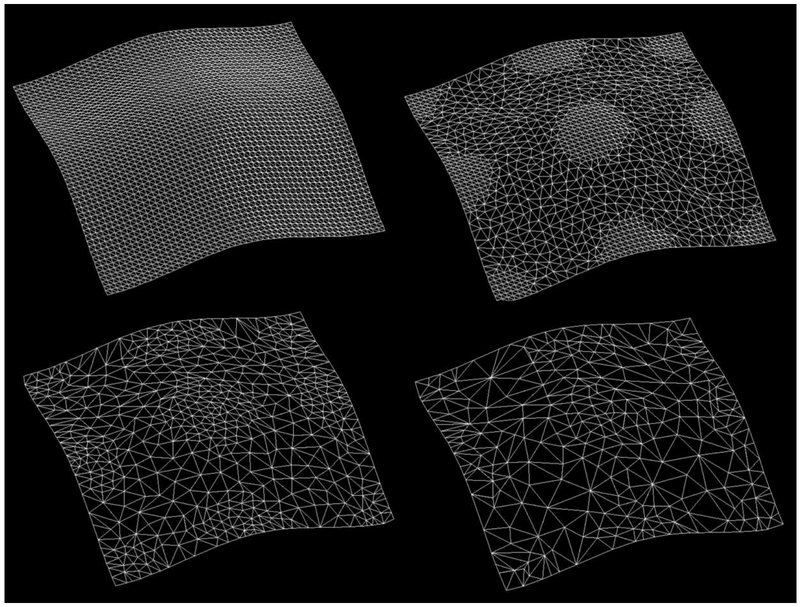

Fig. 5.9. Triangle reduction of 50%, 80%, and 90%
for the graph of $f(x,y) = .1\ \big(\cos(\pi x) + \cos(\pi y)\big),\ x, y \in [-1, 1]$.

Figure 5.10. shows the reduction algorithm applied to a torus and Figure 5.11. shows the original triangular approximation to a human skull (left, about 60,000 triangles obtained by computing a triangular approximation of a particular contour level of a CAT scan data set) and the result after a reduction by 90% (right). All triangles are flat-shaded.

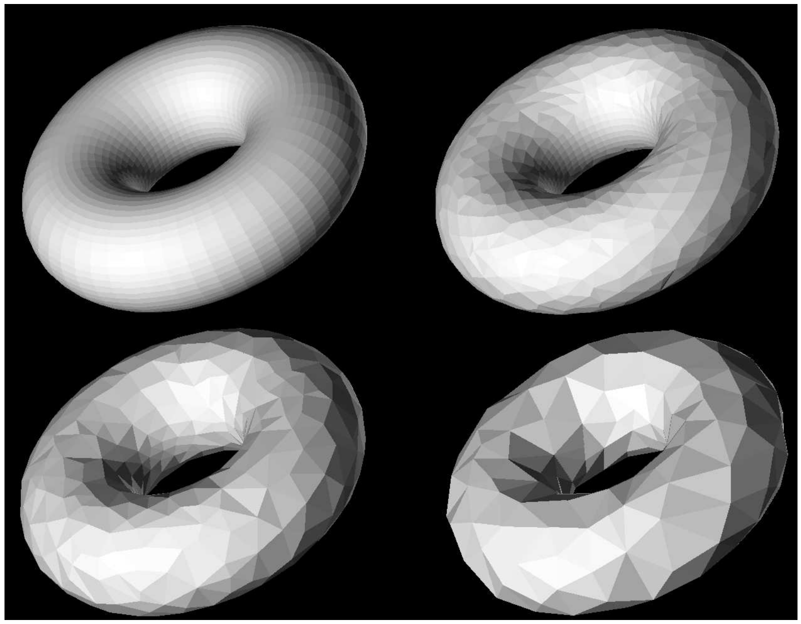

Fig. 5.10. Triangle reduction of 50%, 80%, and 90% for the torus $\left((2+\cos u)\cos v, (2+\cos u)\sin v, \sin u \right)^T$, $u, v \in [0, 2\pi]$.

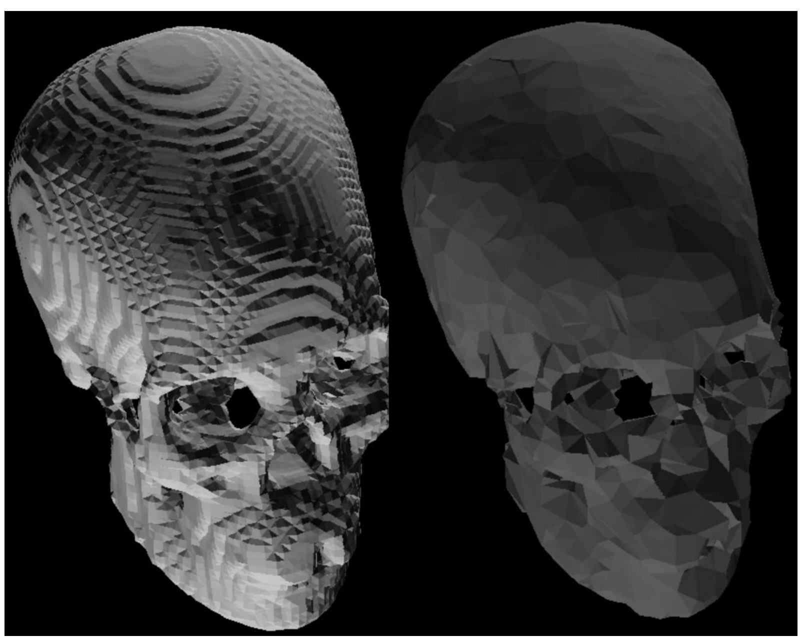

Fig. 5.11. Triangle reduction of 90% for a piecewise triangular approximation of a human skull.

The triangle reduction strategy is tested for graphs of the same bivariate functions as used in chapter 4.2., Table 4.1. Again, all test functions $f(x,y)$ are defined over $[-1,1]\times[-1,1]$ and evaluated on a $51\cdot 51-$grid with equidistant spacing,

$$(x_i,y_j)^T \;=\; \Big(\;-1+\frac{i}{25},-1+\frac{j}{25}\;\Big)^T, \qquad\qquad i,j=0...50,$$

determining points on their graphs,

$$\Big\{\big(\;x_i,y_j,f(x_i,y_j)\;\big)^T \;|\; i,j=0...50\Big\}.$$

A graph's original triangulation is obtained by splitting each quadrilateral specified by its index quadruple

$$\big(\;(i,j),(i+1,j),(i+1,j+1),(i,j+1)\;\big)$$

into the two triangles $T^1_{i,j}$ and $T^2_{i,j}$ identified by their index triples,

$$T^1_{i,j} \;=\; \big(\,(i,j),(i{+}1,j),(i{+}1,j{+}1)\,\big) \qquad \text{and} \qquad T^2_{i,j} \;=\; \big(\,(i,j),(i{+}1,j{+}1),(i,j{+}1)\,\big).$$

The initial triangulation is now reduced using the new technique. Determining a piecewise linear function, original and reduced triangulation are compared at the given knots, (x_i,y_j), $i,j=0...50$. Therefore, the root-mean-square error is

$$\sqrt{\frac{1}{51}\,\frac{1}{51}\,\sum_{j=0}^{50}\,\sum_{i=0}^{50}\,\big(f(x_i,y_j)-\hat{f}(x_i,y_j)\big)^2}, \qquad (5.41.)$$

where $\hat{f}$ denotes the piecewise linear function implied by the reduced triangulation.

In the next table, original and reduced triangulation are compared for different reduction rates, i.e., the original number of triangles is reduced by 50%, 80%, and 90%. Newly constructed triangles can not be removed from an intermediate triangulation in the next iteration step, unless there is no other choice. During the reduction process it is ensured that the final reduced triangulation still covers the whole domain $[-1,1] \times [-1,1]$.

Tab. 5.1. RMS errors of triangle reduction for graphs of bivariate functions.

Function	50%	80%	90%
1. Plane: $.2\ (x+y)$.	0	0	0
2. Cylinder: $\sqrt{2-x^2}$.	.000485	.000999	.002105
3. Sphere: $\sqrt{4-(x^2+y^2)}$.	.000445	.001234	.002288
4. Paraboloid: $.4\ (x^2+y^2)$.	.000610	.001591	.003619
5. Hyperboloid: $.4\ (x^2-y^2)$.	.000248	.000794	.002009
6. Monkey saddle: $.2\ (x^3-3xy^2)$.	.000355	.000843	.001853
7. Cubic polynomial: $.15\ (x^3+2x^2y-xy+2y^2)$.	.000381	.001029	.002160
8. Exponential function: $e^{-\frac{1}{2}(x^2+y^2)}$.	.000336	.000916	.002075
9. Trigonometric function: $.1\ \big(\cos(\pi x)+\cos(\pi y)\big)$.	.000359	.001088	.002054