Fourier Analysis

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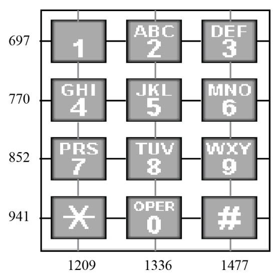
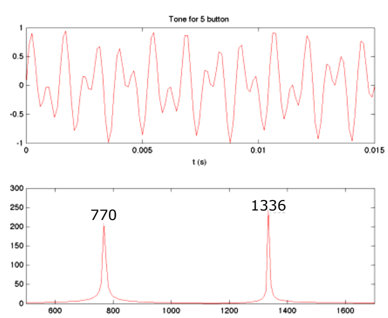
Fourier analysis: the dial tone phone

We use Fourier analysis everyday…without knowing it! A dial tone phone is probably the best example:



Fourier analysis: the dial tone phone



Fourier Analysis

- Fourier series for periodic functions
- Fourier Transform for continuous functions
- Sampling
- Discrete Fourier Transform for discrete functions

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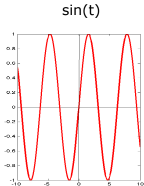
Periodic functions

A function f is periodic, with period T if and only if:

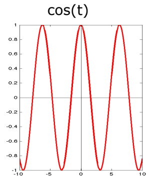
$$\forall x, \quad f(x+T) = f(x)$$

Examples of periodic functions:

sin(t)



cos(t)



Fourier series

A Fourier series of a periodic function f (with period 2π) defined as an expansion of f in a series of sines and cosines such as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx).$$

Fourier series are named in honor of Joseph Fourier (1768-1830), who made important contributions to the study of trigonometric series.

Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx).$$

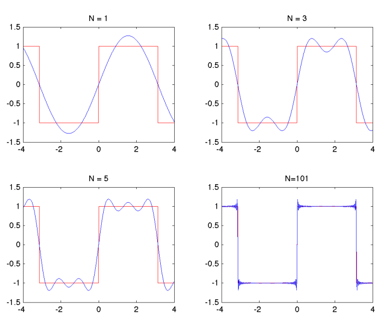
Computing the coefficients a and b:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx,$$

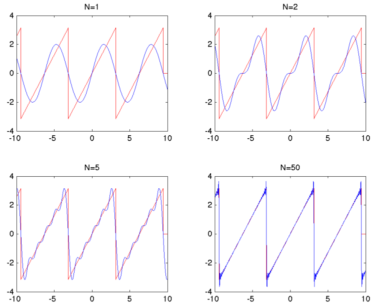
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx,$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx,$$

Fourier series: example 1



Fourier series: example 2



Fourier series

$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

If we express *cos nx* and *sin nx* in exponential form,

$$\cos nx = \frac{1}{2}\left(e^{inx} + e^{-inx}\right), \qquad \sin nx = \frac{1}{2i}\left(e^{inx} - e^{-inx}\right)$$

we may rewrite this equation as

$$g(x) = \sum_{n=-\infty}^{\infty} G(n) e^{inx}$$

in which

$$G(n) = \frac{1}{2}(a_n - ib_n), \qquad \text{and} \qquad G(0) = \frac{1}{2} a_0.$$

$$G(-n) = \frac{1}{2}(a_n + ib_n), \qquad n > 0,$$

Fourier series

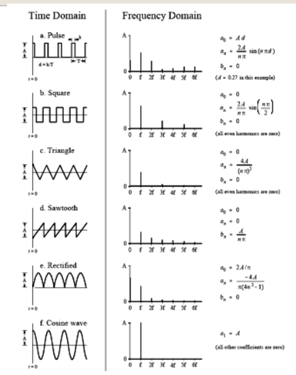
For a function g with period T:

$$g(x) = \sum_{n=-\infty}^{\infty} G(n) e^{i 2\pi \frac{n}{T} x} = \sum_{n=-\infty}^{\infty} G(n) e^{i 2\pi n f_0 x}$$

where f_0 = 1/T is the fundamental frequency for the function g.

In this formula, G(n) can be written as:

$$G(n) = \frac{1}{T} \int_0^T g(x) e^{-i 2\pi n f_0 x} dx$$



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Fourier transform

For a periodic function f with period T, the Fourier coefficients F(n) are computed at multiples nf_0 of a fundamental frequency $f_0=1/T$

For a non periodic function g(t), the Fourier coefficients become a continuous function of the frequencies f:

$$G(f) = \int_{-\infty}^{+\infty} g(t) e^{i2\pi ft}\, dt \qquad (1)$$

g(t) can then be reconstructed according to:

$$g(t) = \int_{-\infty}^{+\infty} G(f) e^{-i2\pi ft}\, df \qquad (2)$$

(1)Is referred to as the **Fourier transform**, while (2) is **the inverse Fourier transform**

Fourier transform

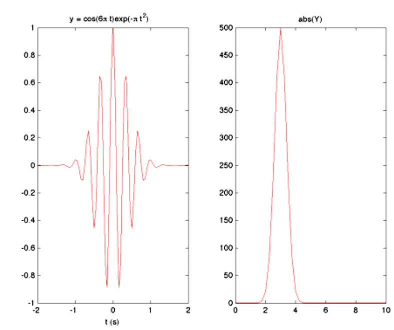
Notes:

-The function g(t) must be integrable; it can be real or complex

-The equations above can be obtained by looking at the limits of the Fourier series

-The Fourier Transform can be rewritten as a function of $\omega = 2\pi f$, the angular frequency.

Fourier transform: example



Properties of the Fourier Transform

1. Superposition (a_1 and a_2 arbitrary constants)	$a_1x_1(t) + a_2x_2(t)$	$a_1X_1(f) + a_2X_2(f)$
2. Time delay	$x(t - t_0)$	$X(f)e^{-j2\pi f t_0}$
3a. Scale change	$x(at)$	$\lvert a\rvert^{-1}X\left(\frac{f}{a}\right)$
b. Time reversal	$x(-t)$	$X(-f) = X * (f)$
4. Duality	$X(t)$	$x(-f)$
5a. Frequency translation	$x(t)e^{j\omega_0 t}$	$X(f - f_0)$
b. Modulation	$x(t)\cos\omega_0 t$	$\frac{1}{2}X(f - f_0) + \frac{1}{2}X(f + f_0)$
6. Differentiation	$\frac{d^n x(t)}{dt^n}$	$(j2\pi f)^n X(f)$
7. Integration	$\int_{-\infty}^{t} x(t')\,dt'$	$(j2\pi f)^{-1}X(f) + \frac{1}{2}X(0)\delta(f)$
8. Convolution	$\int_{-\infty}^{\infty} x_1(t - t')x_2(t')\,dt' = \int_{-\infty}^{\infty} x_1(t')x_2(t - t')\,dt'$	$X_1(f)X_2(f)$
9. Multiplication	$x_1(t)x_2(t)$	$\int_{-\infty}^{\infty} X_1(f - f')X_2(f')\,df' = \int_{-\infty}^{\infty} X_1(f')X_2(f - f')\,df'$

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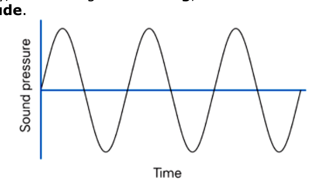
Digital Sound

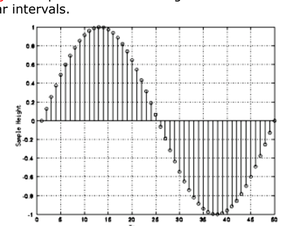
Sound is produced by the vibration of a media like air or water. Audio refers to the sound within the range of human hearing.
Naturally, a sound signal is **analog**, i.e. **continuous in both time and amplitude**.



To store and process sound information in a computer or to transmit it through a computer network, we must first convert the analog signal to **digital form** using an analog-to-digital converter (ADC); the conversion involves two steps: (1) **sampling**, and (2) **quantization**.

Sampling

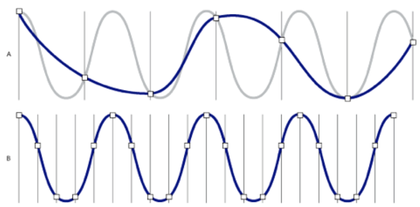
Sampling is the process of examining the value of a continuous function at regular intervals.



Sampling usually occurs at uniform intervals, which are referred to as sampling intervals. The reciprocal of sampling interval is referred to as the sampling frequency or sampling rate.
If the sampling is done in time domain, the unit of sampling interval is second and the unit of sampling rate is Hz, which means cycles per second.

Sampling

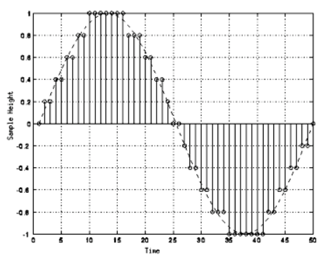
Note that choosing the sampling rate is not innocent:



A higher sampling rate usually allows for a better representation of the original sound wave. However, when the sampling rate is set to twice the highest frequency in the signal, the original sound wave can be reconstructed without loss from the samples. This is known as the Nyquist theorem.

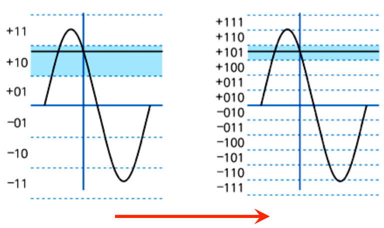
Quantization

Quantization is the process of limiting the value of a sample of a continuous function to one of a predetermined number of allowed values, which can then be represented by a finite number of bits.



Quantization

The number of bits used to store each intensity defines the accuracy of the digital sound:



Adding one bit makes the sample twice as accurate

Audio Sound

Sampling:

The human ear can hear sound up to 20,000 Hz: a sampling rate of 40,000 Hz is therefore sufficient. The standard for digital audio is 44,100 Hz.

Quantization:

The current standard for the digital representation of audio sound is to use 16 bits (i.e 65536 levels, half positive and half negative)

How much space do we need to store one minute of music?

- 60 seconds
- 44,100 samples
- 16 bits (2 bytes) per sample
- 2 channels (stereo)

S = 60x44100x2x2 = 10,534,000 bytes ≈ 10 MB !!
1 hour of music would be more than 600 MB !

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Discrete time Fourier Transform

Given a discrete set of values x(n), with n integer; the discrete Time Fourier transform of x is:

$$X(f) = \sum_{n=-\infty}^{n=+\infty} x(n) e^{i2\pi fn}$$

Notice that X(f) is periodic:

$$X(f+k) = \sum_{n=-\infty}^{n=+\infty} x(n) e^{i2\pi(f+k)n} = \sum_{n=-\infty}^{n=+\infty} x(n) e^{i2\pi fn} e^{i2\pi n} = X(f)$$

Discrete Fourier Transform

The sequence of numbers $x_0,\ldots x_{N-1}$ is transformed into a new series of numbers $X_0,\ldots.X_{N-1}$ according to the digital Fourier transform (DFT) formula:

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{i2\pi \frac{kn}{N}}$$

The inverse DFT is given by:

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{-i2\pi \frac{kn}{N}}$$

Discrete Fourier Transform

Notes:

-If x(n) is a time signal, and Δ is the constant time interval between two time points, then the total duration of the time signal is (N-1)*Δ; the fundamental frequency is f0=1/(N*Δ)

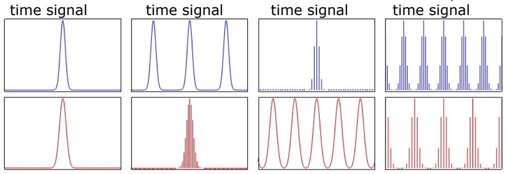
-If n is a power of 2, X(k) can be computed really fast using the Fast Fourier Transform (FFT)
The corresponding command in Matlab is:

X = fft(x)

-x(n) can be real or complex. X(k) is always complex.

Fourier analysis

Continuous time signal | Periodic time signal | Discrete time signal | Discrete, finite time signal



Continuous Fourier domain | Discrete Fourier domain | Periodic Fourier domain | Discrete, finite Fourier domain

Summary table

Time Duration		
Finite	Infinite	
Discrete FT (DFT) $X(k) = \sum_{n=0}^{N-1} x(n)e^{-j\omega_k n}$ $k = 0, 1, \ldots, N-1$	Discrete Time FT (DTFT) $X(\omega) = \sum_{n=-\infty}^{+\infty} x(n)e^{-j\omega n}$ $\omega \in [-\pi, +\pi)$	discr. time n
Fourier Series (FS) $X(k) = \frac{1}{P}\int_0^P x(t)e^{-j\omega_k t}dt$ $k = -\infty, \ldots, +\infty$	Fourier Transform (FT) $X(\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$ $\omega \in (-\infty, +\infty)$	cont. time t
discrete freq. k	continuous freq. ω	