Fourier Analysis

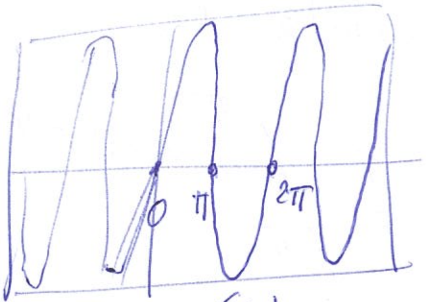
I) Fourier series of periodic functions

I.1) Periodic functions

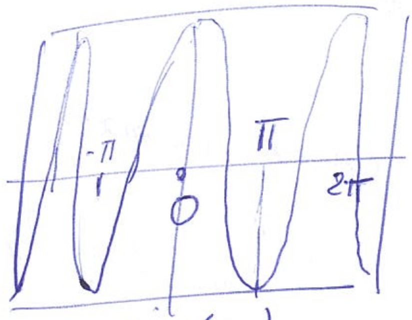
A function f from $\mathbb{R}$ to $\mathbb{R}$ is periodic of period T if and only if:

$$\forall x, \quad f(x+T) = f(x)$$

Examples:



sin(t)



cos(t)

Note that: if T is a period, kT is also a period. The period is the smallest T that satisfies the equation above.

I.2) Fourier series of a function with period 2π

Any function f periodic of period 2π, can be written as an (infinite) series of sines and cosines:

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right)}$$

This expansion is called a Fourier series, in honor of Joseph Fourier (1768-1830) who made important contributions to the study of trigonometric series

Computing the coefficients:

First, let us note that:

$$I = \int_{-\pi}^{\pi} \cos nx \sin mx \, dx = 0$$

Indeed:

$$\cos nx \sin mx = \frac{1}{2} \sin(n+m)x + \frac{1}{2} \sin(m-n)x$$

$$I = \frac{1}{2} \int_{-\pi}^{\pi} \sin(n+m)x \, dx + \frac{1}{2} \int_{-\pi}^{\pi} \sin(m-n)x \, dx$$

$$\int_{-\pi}^{\pi} \sin(n+m)x \, dx = -\left[\frac{\cos(n+m)x}{(n+m)}\right]_{-\pi}^{\pi} = 0$$

$$\int_{-\pi}^{\pi} \sin(n-m)x \, dx = -\left[\frac{\cos(n-m)x}{(n-m)}\right]_{-\pi}^{\pi} = 0 \quad \text{if } n \neq m$$

$$= \int_{-\pi}^{\pi} 0 \, dx = 0 \quad \text{if } n = m$$

Similarly:

$$J = \int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \begin{cases} 0 & \text{if } n \neq m \\ \pi & \text{if } n = m \end{cases}$$

$$K = \int_{-\pi}^{\pi} \sin nx \sin mx \, dx = \begin{cases} 0 & \text{if } n \neq m \\ \pi & \text{if } n = m \end{cases}$$

Remember that:

$$f(x) = a_0 + \sum_{n=1}^{+\infty} a_n \cos nx + b_n \sin nx$$

Then:

$$\int_{-\pi}^{\pi} f(x) \cos px \, dx = a_0 \int_{-\pi}^{\pi} \cos px \, dx + \sum_{n=1}^{+\infty} a_n \int_{-\pi}^{\pi} \cos nx \cos px \, dx + b_n \int_{-\pi}^{\pi} \sin nx \cos px \, dx$$

$$= 0 + a_p \pi + 0$$

therefore:

$$\boxed{a_p = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos px \, dx}$$

Similarly:

$$\boxed{b_p = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin px \, dx}$$

And

$$\boxed{a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx}$$

Example

$$f(x) = \begin{cases} -1 & -\pi \leq x \leq 0 \\ 1 & 0 \leq x \leq \pi \end{cases}$$

periodic, of period 2π

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} -dx + \frac{1}{\pi}\int_{0}^{\pi} dx$$

$$a_0 = 0$$

$$a_p = \frac{1}{\pi}\int_{-\pi}^{0} -\cos px\,dx + \frac{1}{\pi}\int_{0}^{\pi} \cos px\,dx$$

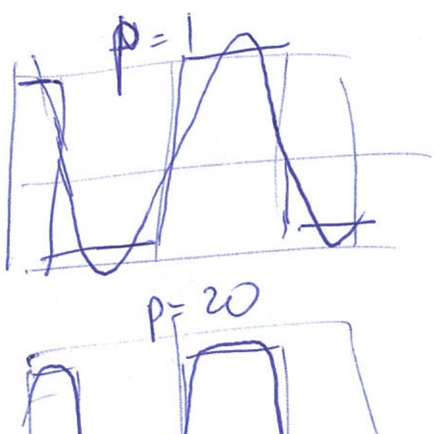
$$= \frac{1}{\pi}\left(\int_{\pi}^{0} \cos px\,dx + \int_{0}^{\pi} \cos px\,dx\right) = 0$$

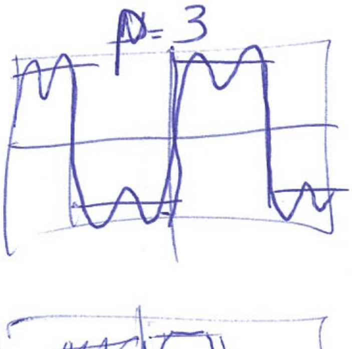
$$b_p = \frac{1}{\pi}\int_{-\pi}^{0} -\sin px\,dx + \frac{1}{\pi}\int_{0}^{\pi} \sin px\,dx$$

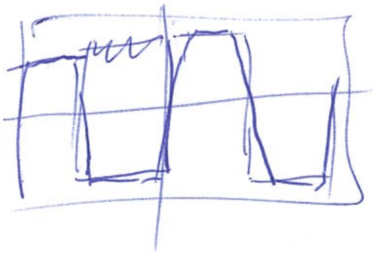
$$= \frac{2}{\pi}\int_{0}^{\pi} \sin px\,dx$$

$$b_p = \frac{2}{\pi}\left[\frac{-\cos px}{p}\right]_{0}^{\pi} = \frac{2}{\pi}\left(1 - \frac{(-1)^p}{p}\right)$$

$$f(x) = \frac{2}{\pi}\sum_{p=1}^{+\infty}\left(1 - \frac{(-1)^p}{p}\right)\sin px$$







I.3) Fourier series for function with period T

Let $g(x)$ be a function with period T:

$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} \left(a_n \cos \frac{2\pi}{T} n x + b_n \sin \frac{2\pi}{T} n x \right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{+\infty} a_n \cos(2\pi n f_0 x) + b_n \sin(2\pi n f_0 x)$$

where $f_0 = \frac{1}{T}$

Complex representation.

Euler's formula: $e^{inx} = \cos nx + i \sin nx$

Therefore:

$$\cos nx = \frac{e^{inx} + e^{-inx}}{2}$$

$$\sin nx = \frac{e^{inx} - e^{-inx}}{2i}$$

Defining:

$$c_n = \frac{1}{2}(a_n - i b_n)$$

$$c_{-n} = \frac{1}{2}(a_n + i b_n)$$

$$c_0 = \frac{1}{2} a_0$$

We get:

$$\boxed{g(x) = \sum_{n=-\infty}^{+\infty} c_n e^{i 2\pi n f_0 x}}$$

II. Fourier transform for continuous functions

For a periodic function f with period T, the Fourier coefficients* a_n, b_n (or c_n) are computed at multiples $n f_0$ of a fundamental frequency $f_0 = \frac{1}{T}$

For a non periodic function $g(t)$, the Fourier coefficients become a continuous function of the frequencies f:

$$C(f) = \int_{-\infty}^{+\infty} g(t)\, e^{i 2\pi f t}\, dt$$

$g(t)$ is then reconstructed according to:

$$g(t) = \int_{-\infty}^{+\infty} C(f)\, e^{-i 2\pi f t}\, df$$

$C(f)$ refers to the Fourier transform of g, while $g(t)$ is the inverse Fourier transform of $C(f)$.

Notes:

(1) The function $g(t)$ must be integrable over the whole domain. It can be real or complex

(2) The equations above can be understood as limits of the Fourier series for $T \to +\infty$

(3) The Fourier transform can be written as a function of frequencies, f, or as function of angular frequencies, ω, with $\omega = 2\pi f$

$$C(\omega) = \int_{-\infty}^{+\infty} g(t)\, e^{i\omega t}\, dt$$

III. Sampling

Let us consider a signal $s(t)$, where t can be time. Naturally, $s(t)$ is analog, i.e continuous in both time and amplitude.

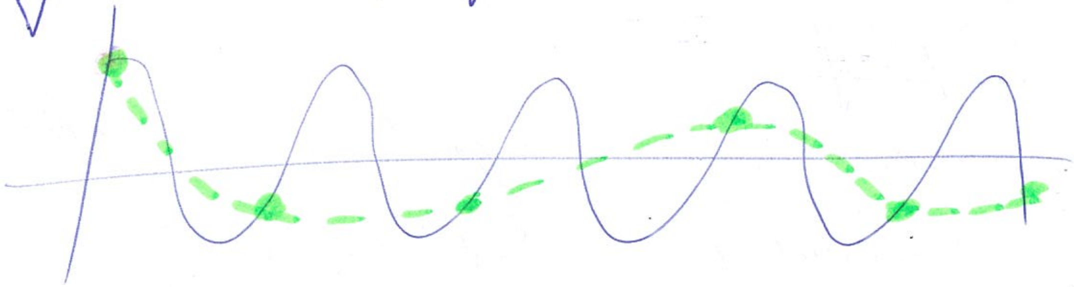
To process and store $s(t)$, or to transmit it, we must first convert the analog signal to digital form. The conversion involves two steps:

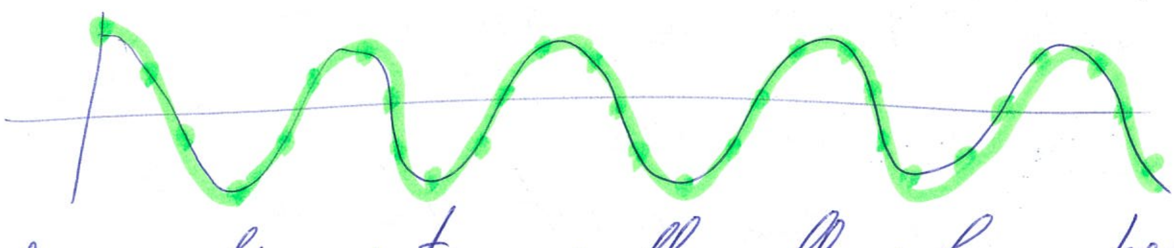
(i) sampling

(ii) quantization

III.1. Sampling

Sampling is the process of collecting the value of a continuous function at usually regular intervals, which are refered to as sampling intervals. The reciprocal of the sampling interval is called the sampling frequency a sampling rate. If the sampling is done in time domain, the unit of sampling interval is second and the unit of sampling rate is Hz, which means cycles per second. Choosing the sampling rate is not innocent:

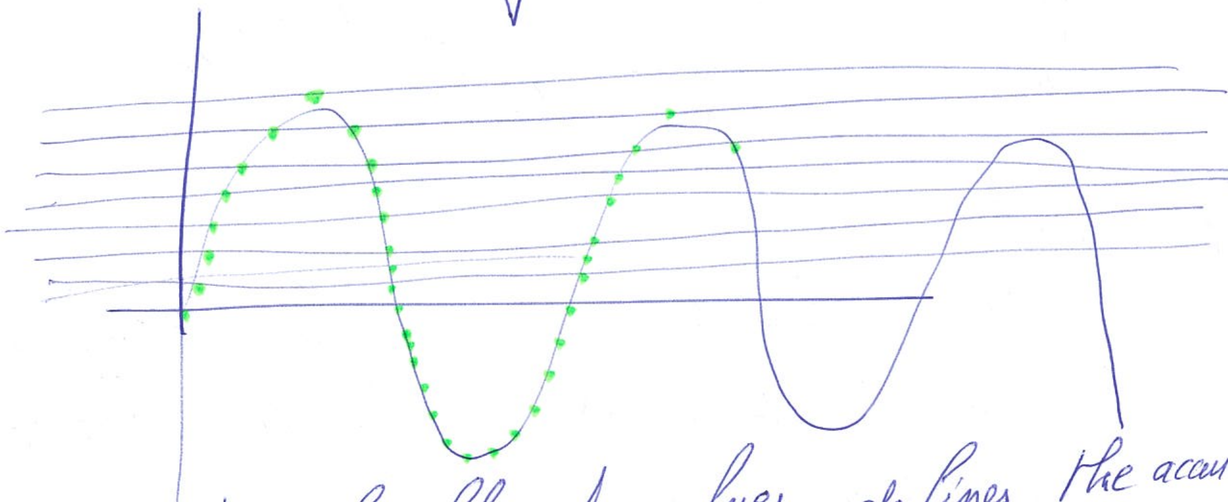




A higher sampling rate usually allows for a better representation of the original sound wave. However, when the sampling rate is set to more than twice the highest frequency in the signal, this signal can be reconstructed without loss. This is known as the Nyquist theorem.

III.2. Quantization

Quantization is the process of limiting the value of a sample of a continuous function to one of a predetermined number of allowed values.



The number of allowed values defines the accuracy of the digital signal.

IV Discrete Fourier transform for discrete functions

Let us consider a set of numbers

$x_0, \ldots x_{N-1}$, corresponding to the sampling of a function $f(t)$ with time interval Δ. The total duration of the sampled signal is $(N-1)\Delta$. It is assumed to be periodic, of period $N\Delta$. Its fundamental frequency is $f_0 = \dfrac{1}{N\Delta}$.

The discrete Fourier transform (DFT) of the signal $\{x_0, \ldots\ldots x_{N-1}\}$ is defined as:

$$X_k = \sum_{n=0}^{N-1} x_n e^{-i2\pi \frac{kn}{N}}$$

Note that:

(1) This formula is independent of the time step Δ, or of the fundamental frequency f_0.
X_k is the Fourier coefficient at frequency k, (which really means at frequency kf_0)

(2) X_k is periodic, of period N:

$$X_{k+N} = \sum_{n=0}^{N-1} x_n e^{-\frac{i2\pi(k+N)n}{N}}$$

$$= \sum_{n=0}^{N-1} x_n e^{-\frac{i2\pi kn}{N}} \underbrace{e^{-i2\pi n}}_{1}$$

$$= X_k$$

(3) If x_n is real, $X_{N-k} = \overline{X_k}$

$$X_{N-k} = \sum_{n=0}^{N-1} x_n e^{-i2\pi n} e^{\frac{i2\pi kn}{N}} = \sum_{n=0}^{N-1} x_n e^{\frac{i2\pi kn}{N}}$$

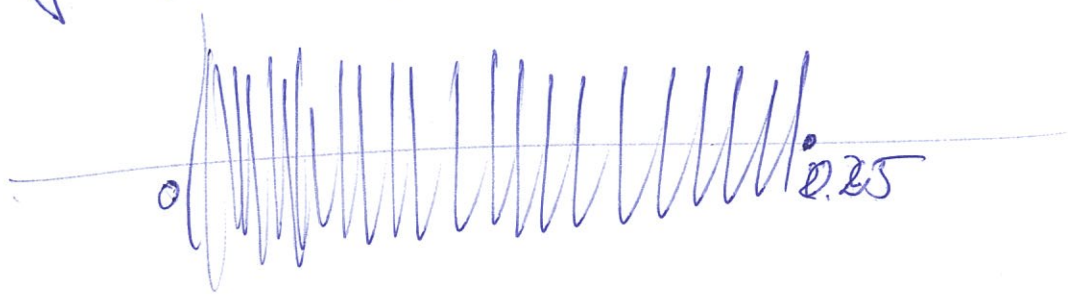
$$= \overline{X_k}$$

All the information is in $\left[0, \frac{N}{2}\right]$

Example:

- $\Delta = \frac{1}{8192}$: sampling.
- $t = 0 : \Delta : 0.25$: time interval $[0, 0.25s]$
sampled each Δ second.
- $f_1 = 770$; $f_2 = 1209$: two frequencies.
- Build signal y as sum of sinusoids with frequencies f_1 and f_2:

$$y = (\sin 2\pi f_1 t + + \sin 2\pi f_2 t)/2$$



- Compute DFT:

$z = fft(y);$

Note: z computed at $0, .. k, \dots N-1$ where
$N = size(t) = 2049$.
k corresponds to $k f_0$, where $f_0 = \frac{1}{N\Delta}$

- $f_0 = \frac{1}{max(size(t)) * \Delta}$
- $f = 0 : f_0 : (N-1) f_0$

plot(f, abs(z))