3 are relatively prime to 10. Therefore the sum can no longer be 0 modulo 10. **45.** Working modulo 10, solve for d_9. The check digit for 11100002 is 5. **47.** PLEASE SEND MONEY **49. a)** QAL HUVEM AT WVESGB **b)** QXB EVZZL ZEVZZRFS

CHAPTER 5

Section 5.1

1. Let $P(n)$ be the statement that the train stops at station n. *Basis step:* We are told that $P(1)$ is true. *Inductive step:* We are told that $P(n)$ implies $P(n+1)$ for each $n \geq 1$. Therefore by the principle of mathematical induction, $P(n)$ is true for all positive integers n. **3. a)** $1^2 = 1 \cdot 2 \cdot 3/6$ **b)** Both sides of $P(1)$ shown in part (a) equal 1. **c)** $1^2 + 2^2 + \cdots + k^2 = k(k+1)(2k+1)/6$ **d)** For each $k \geq 1$ that $P(k)$ implies $P(k+1)$; in other words, that assuming the inductive hypothesis [see part (c)] we can show $1^2 + 2^2 + \cdots + k^2 + (k+1)^2 = (k+1)(k+2)(2k+3)/6$ **e)** $(1^2 + 2^2 + \cdots + k^2) + (k+1)^2 = [k(k+1)(2k+1)/6] + (k+1)^2 = [(k+1)/6][k(2k+1) + 6(k+1)] = [(k+1)/6](2k^2+7k+6) = [(k+1)/6](k+2)(2k+3) = (k+1)(k+2)(2k+3)/6$ **f)** We have completed both the basis step and the inductive step, so by the principle of mathematical induction, the statement is true for every positive integer n. **5.** Let $P(n)$ be "$1^2 + 3^2 + \cdots + (2n+1)^2 = (n+1)(2n+1)(2n+3)/3$." *Basis step:* $P(0)$ is true because $1^2 = 1 = (0+1)(2\cdot 0+1)(2\cdot 0+3)/3$. *Inductive step:* Assume that $P(k)$ is true. Then $1^2+3^2+\cdots+(2k+1)^2+[2(k+1)+1]^2 = (k+1)(2k+1)(2k+3)/3+(2k+3)^2 = (2k+3)[(k+1)(2k+1)/3+(2k+3)] = (2k+3)(2k^2+9k+10)/3 = (2k+3)(2k+5)(k+2)/3 = [(k+1)+1][2(k+1)+1][2(k+1)+3]/3$. **7.** Let $P(n)$ be "$\sum_{j=0}^{n} 3\cdot 5^j = 3(5^{n+1}-1)/4$." *Basis step:* $P(0)$ is true because $\sum_{j=0}^{0} 3\cdot 5^j = 3 = 3(5^1-1)/4$. *Inductive step:* Assume that $\sum_{j=0}^{k} 3\cdot 5^j = 3(5^{k+1}-1)/4$. Then $\sum_{j=0}^{k+1} 3\cdot 5^j = (\sum_{j=0}^{k} 3\cdot 5^j) + 3\cdot 5^{k+1} = 3(5^{k+1}-1)/4 + 3\cdot 5^{k+1} = 3(5^{k+1}+4\cdot 5^{k+1}-1)/4 = 3(5^{k+2}-1)/4$. **9. a)** $2+4+6+\cdots+2n = n(n+1)$ **b)** *Basis step:* $2 = 1\cdot(1+1)$ is true. *Inductive step:* Assume that $2+4+6+\cdots+2k = k(k+1)$. Then $(2+4+6+\cdots+2k) + 2(k+1) = k(k+1)+2(k+1) = (k+1)(k+2)$. **11. a)** $\sum_{j=1}^{n} 1/2^j = (2^n-1)/2^n$ **b)** *Basis step:* $P(1)$ is true because $\frac{1}{2} = (2^1-1)/2^1$. *Inductive step:* Assume that $\sum_{j=1}^{k} 1/2^j = (2^k-1)/2^k$. Then $\sum_{j=1}^{k+1} \frac{1}{2^j} = (\sum_{j=1}^{k} \frac{1}{2^j}) + \frac{1}{2^{k+1}} = \frac{2^k-1}{2^k} + \frac{1}{2^{k+1}} = \frac{2^{k+1}-2+1}{2^{k+1}} = \frac{2^{k+1}-1}{2^{k+1}}$. **13.** Let $P(n)$ be "$1^2 - 2^2 + 3^2 - \cdots + (-1)^{n-1}n^2 = (-1)^{n-1}n(n+1)/2$." *Basis step:* $P(1)$ is true because $1^2 = 1 = (-1)^0 1^2$. *Inductive step:* Assume that $P(k)$ is true. Then $1^2 - 2^2 + 3^2 - \cdots + (-1)^{k-1}k^2 + (-1)^k(k+1)^2 = (-1)^{k-1}k(k+1)/2 + (-1)^k(k+1)^2 = (-1)^k(k+1)[-k/2+(k+1)] = (-1)^k(k+1)[(k/2)+1] = (-1)^k(k+1)(k+2)/2$. **15.** Let $P(n)$ be "$1\cdot 2+2\cdot 3+\cdots+n(n+1) = n(n+1)(n+2)/3$." *Basis step:* $P(1)$ is true because $1\cdot 2 = 2 = 1(1+1)(1+2)/3$. *Inductive step:* Assume that $P(k)$ is true. Then $1\cdot 2+2\cdot 3+\cdots+k(k+1)+(k+1)(k+2) = [k(k+1)(k+2)/3]+(k+1)(k+2) = (k+1)(k+2)[(k/3)+1] = (k+1)(k+2)(k+3)/3$. **17.** Let $P(n)$ be the statement that $1^4+2^4+3^4+\cdots+n^4 = n(n+1)(2n+1)(3n^2+3n-1)/30$. $P(1)$ is true because $1\cdot 2\cdot 3\cdot 5/30 = 1$. Assume that $P(k)$ is true. Then $(1^4+2^4+3^4+\cdots+k^4)+(k+1)^4 = k(k+1)(2k+1)(3k^2+3k-1)/30+(k+1)^4 = [(k+1)/30][k(2k+1)(3k^2+3k-1)+30(k+1)^3] = [(k+1)/30](6k^4+39k^3+91k^2+89k+30) = [(k+1)/30](k+2)(2k+3)[3(k+1)^2+3(k+1)-1]$. This demonstrates that $P(k+1)$ is true. **19. a)** $1+\frac{1}{4} < 2-\frac{1}{2}$ **b)** This is true because 5/4 is less than 6/4. **c)** $1+\frac{1}{4}+\cdots+\frac{1}{k^2} < 2-\frac{1}{k}$ **d)** For each $k \geq 2$ that $P(k)$ implies $P(k+1)$; in other words, we want to show that assuming the inductive hypothesis [see part (c)] we can show $1+\frac{1}{4}+\cdots+\frac{1}{k^2}+\frac{1}{(k+1)^2} < 2-\frac{1}{k+1}$ **e)** $1+\frac{1}{4}+\cdots+\frac{1}{k^2}+\frac{1}{(k+1)^2} < 2-\frac{1}{k}+\frac{1}{(k+1)^2} = 2-\left[\frac{1}{k}-\frac{1}{(k+1)^2}\right] = 2-\left[\frac{k^2+2k+1-k}{k(k+1)^2}\right] = 2-\frac{k^2+k}{k(k+1)^2}-\frac{1}{k(k+1)^2} = 2-\frac{1}{k+1}-\frac{1}{k(k+1)^2} < 2-\frac{1}{k+1}$ **f)** We have completed both the basis step and the inductive step, so by the principle of mathematical induction, the statement is true for every integer n greater than 1. **21.** Let $P(n)$ be "$2^n > n^2$." *Basis step:* $P(5)$ is true because $2^5 = 32 > 25 = 5^2$. *Inductive step:* Assume that $P(k)$ is true, that is, $2^k > k^2$. Then $2^{k+1} = 2\cdot 2^k > k^2+k^2 > k^2+4k \geq k^2+2k+1 = (k+1)^2$ because $k > 4$. **23.** By inspection we find that the inequality $2n+3 \leq 2^n$ does not hold for $n = 0, 1, 2, 3$. Let $P(n)$ be the proposition that this inequality holds for the positive integer n. $P(4)$, the basis case, is true because $2\cdot 4+3 = 11 \leq 16 = 2^4$. For the inductive step assume that $P(k)$ is true. Then, by the inductive hypothesis, $2(k+1)+3 = (2k+3)+2 < 2^k+2$. But because $k \geq 1$, $2^k+2 \leq 2^k+2^k = 2^{k+1}$. This shows that $P(k+1)$ is true. **25.** Let $P(n)$ be "$1+nh \leq (1+h)^n$, $h > -1$." *Basis step:* $P(0)$ is true because $1+0\cdot h = 1 \leq 1 = (1+h)^0$. *Inductive step:* Assume $1+kh \leq (1+h)^k$. Then because $(1+h) > 0$, $(1+h)^{k+1} = (1+h)(1+h)^k \geq (1+h)(1+kh) = 1+(k+1)h+kh^2 \geq 1+(k+1)h$. **27.** Let $P(n)$ be "$1/\sqrt{1}+1/\sqrt{2}+1/\sqrt{3}+\cdots+1/\sqrt{n} > 2\left(\sqrt{n+1}-1\right)$." *Basis step:* $P(1)$ is true because $1 > 2\left(\sqrt{2}-1\right)$. *Inductive step:* Assume that $P(k)$ is true. Then $1+1/\sqrt{2}+\cdots+1/\sqrt{k}+1/\sqrt{k+1} > 2\left(\sqrt{k+1}-1\right)+1/\sqrt{k+1}$. If we show that $2\left(\sqrt{k+1}-1\right)+1/\sqrt{k+1} > 2\left(\sqrt{k+2}-1\right)$, it follows that $P(k+1)$ is true. This inequality is equivalent to $2\left(\sqrt{k+2}-\sqrt{k+1}\right) < 1/\sqrt{k+1}$, which is equivalent to $2\left(\sqrt{k+2}-\sqrt{k+1}\right)\left(\sqrt{k+2}+\sqrt{k+1}\right) < \sqrt{k+1}/\sqrt{k+1}+\sqrt{k+2}/\sqrt{k+1}$. This is equivalent to $2 < 1+\sqrt{k+2}/\sqrt{k+1}$, which is clearly true. **29.** Let $P(n)$ be "$H_{2^n} \leq 1+n$." *Basis step:* $P(0)$ is true because $H_{2^0} = H_1 = 1 \leq 1+0$. *Inductive step:* Assume that $H_{2^k} \leq 1+k$. Then $H_{2^{k+1}} = H_{2^k}+\sum_{j=2^k+1}^{2^{k+1}} \frac{1}{j} \leq 1+k+2^k\left(\frac{1}{2^{k+1}}\right) < 1+k+1 = 1+(k+1)$. **31.** *Basis step:* $1^2+1 = 2$ is divisible by 2. *Inductive step:* Assume the inductive hypothesis, that k^2+k is divisible by 2. Then $(k+1)^2+(k+1) = k^2+2k+1+k+1 = (k^2+k)+2(k+1)$, the sum of a multiple of 2 (by the inductive hypothesis) and a multiple of 2 (by definition), hence, divisible by 2. **33.** Let $P(n)$ be "$n^5 - n$ is divisible by 5." *Basis step:* $P(0)$ is true because $0^5 - 0 = 0$ is divisible by 5. *Inductive step:* Assume that $P(k)$ is true, that is, $k^5 - 5$ is divisible by 5. Then $(k+1)^5-(k+1) = (k^5+5k^4+10k^3+10k^2+5k+1)-(k+1) = (k^5 - k) + 5(k^4 + 2k^3 + 2k^2 + k)$ is also divisible by 5, because both terms in this sum are divisible by 5. **35.** Let $P(n)$ be the proposition that $(2n - 1)^2 - 1$ is divisible by 8. The basis case $P(1)$ is true because $8 \mid 0$. Now assume that $P(k)$ is true. Because $[(2(k + 1) - 1]^2 - 1 = [(2k - 1)^2 - 1] + 8k$, $P(k + 1)$ is true because both terms on the right-hand side are divisible by 8. This shows that $P(n)$ is true for all positive integers n, so $m^2 - 1$ is divisible by 8 whenever m is an odd positive integer. **37.** *Basis step:* $11^{1+1}+12^{2\cdot 1-1} = 121+12 = 133$ *Inductive step:* Assume the inductive hypothesis, that $11^{n+1} + 12^{2n-1}$ is divisible by 133. Then $11^{(n+1)+1} + 12^{2(n+1)-1} = 11 \cdot 11^{n+1} + 144 \cdot 12^{2n-1} = 11 \cdot 11^{n+1} + (11 + 133) \cdot 12^{2n-1} = 11(11^{n+1} + 12^{2n-1}) + 133 \cdot 12^{2n-1}$. The expression in parentheses is divisible by 133 by the inductive hypothesis, and obviously the second term is divisible by 133, so the entire quantity is divisible by 133, as desired. **39.** *Basis step:* $A_1 \subseteq B_1$ tautologically implies that $\bigcap_{j=1}^{1} A_j \subseteq \bigcap_{j=1}^{1} B_j$. *Inductive step:* Assume the inductive hypothesis that if $A_j \subseteq B_j$ for $j = 1, 2, \ldots, k$, then $\bigcap_{j=1}^{k} A_j \subseteq \bigcap_{j=1}^{k} B_j$. We want to show that if $A_j \subseteq B_j$ for $j = 1, 2, \ldots, k + 1$, then $\bigcap_{j=1}^{k+1} A_j \subseteq \bigcap_{j=1}^{k+1} B_j$. Let x be an arbitrary element of $\bigcap_{j=1}^{k+1} A_j = \left(\bigcap_{j=1}^{k} A_j\right) \cap A_{k+1}$. Because $x \in \bigcap_{j=1}^{k} A_j$, we know by the inductive hypothesis that $x \in \bigcap_{j=1}^{k} B_j$; because $x \in A_{k+1}$, we know from the given fact that $A_{k+1} \subseteq B_{k+1}$ that $x \in B_{k+1}$. Therefore, $x \in \left(\bigcap_{j=1}^{k} B_j\right) \cap B_{k+1} = \bigcap_{j=1}^{k+1} B_j$. **41.** Let $P(n)$ be "$(A_1 \cup A_2 \cup \cdots \cup A_n) \cap B = (A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cup (A_n \cap B)$." *Basis step:* $P(1)$ is trivially true. *Inductive step:* Assume that $P(k)$ is true. Then $(A_1 \cup A_2 \cup \cdots \cup A_k \cup A_{k+1}) \cap B = [(A_1 \cup A_2 \cup \cdots \cup A_k) \cup A_{k+1}] \cap B = [(A_1 \cup A_2 \cup \cdots \cup A_k) \cap B] \cup (A_{k+1} \cap B) = [(A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cup (A_k \cap B)] \cup (A_{k+1} \cap B) = (A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cup (A_k \cap B) \cup (A_{k+1} \cap B)$. **43.** Let $P(n)$ be "$\overline{\bigcup_{k=1}^{n} A_k} = \bigcap_{k=1}^{n} \overline{A_k}$." *Basis step:* $P(1)$ is trivially true. *Inductive step:* Assume that $P(k)$ is true. Then $\overline{\bigcup_{j=1}^{k+1} A_j} = \overline{\left(\bigcup_{j=1}^{k} A_j\right) \cup A_{k+1}} = \overline{\left(\bigcup_{j=1}^{k} A_j\right)} \cap \overline{A_{k+1}} = \left(\bigcap_{j=1}^{k} \overline{A_j}\right) \cap \overline{A_{k+1}} = \bigcap_{j=1}^{k+1} \overline{A_j}$. **45.** Let $P(n)$ be the statement that a set with n elements has $n(n-1)/2$ two-element subsets. $P(2)$, the basis case, is true, because a set with two elements has one subset with two elements—namely, itself—and $2(2 - 1)/2 = 1$. Now assume that $P(k)$ is true. Let S be a set with $k + 1$ elements. Choose an element a in S and let $T = S - \{a\}$. A two-element subset of S either contains a or does not. Those subsets not containing a are the subsets of T with two elements; by the inductive hypothesis there are $k(k-1)/2$ of these. There are k subsets of S with two elements that contain a, because such a subset contains a and one of the k elements in T. Hence, there are $k(k-1)/2+k = (k+1)k/2$ two-element subsets of S. This completes the inductive proof. **47.** Reorder the locations if necessary so that $x_1 \le x_2 \le x_3 \le \cdots \le x_d$. Place the first tower at position $t_1 = x_1 + 1$. Assume tower k has been placed at position t_k. Then place tower $k + 1$ at position $t_{k+1} = x + 1$, where x is the smallest x_i greater than $t_k + 1$. **49.** The two sets do not overlap if $n + 1 = 2$. In fact, the conditional statement $P(1) \to P(2)$ is false. **51.** The mistake is in applying the inductive hypothesis to look at $\max(x - 1, y - 1)$, because even though x and y are positive integers, $x - 1$ and $y - 1$ need not be (one or both could be 0). **53.** For the basis step ($n = 2$) the first person cuts the cake into two portions that she thinks are each $1/2$ of the cake, and the second person chooses the portion he thinks is at least $1/2$ of the cake (at least one of the pieces must satisfy that condition). For the inductive step, suppose there are $k + 1$ people. By the inductive hypothesis, we can suppose that the first k people have divided the cake among themselves so that each person is satisfied that he got at least a fraction $1/k$ of the cake. Each of them now cuts his or her piece into $k+1$ pieces of equal size. The last person gets to choose one piece from each of the first k people's portions. After this is done, each of the first k people is satisfied that she still has $(1/k)(k/(k + 1)) = 1/(k + 1)$ of the cake. To see that the last person is satisfied, suppose that he thought that the ith person ($1 \le i \le k$) had a portion p_i of the cake, where $\sum_{i=1}^{k} p_i = 1$. By choosing what he thinks is the largest piece from each person, he is satisfied that he has at least $\sum_{i=1}^{k} p_i/(k+1) = (1/(k+1)) \sum_{i=1}^{k} p_i = 1/(k+1)$ of the cake. **55.** We use the notation (i, j) to mean the square in row i and column j and use induction on $i + j$ to show that every square can be reached by the knight. *Basis step:* There are six base cases, for the cases when $i + j \le 2$. The knight is already at $(0, 0)$ to start, so the empty sequence of moves reaches that square. To reach $(1, 0)$, the knight moves $(0, 0) \to (2, 1) \to (0, 2) \to (1, 0)$. Similarly, to reach $(0, 1)$, the knight moves $(0, 0) \to (1, 2) \to (2, 0) \to (0, 1)$. Note that the knight has reached $(2, 0)$ and $(0, 2)$ in the process. For the last basis step there is $(0, 0) \to (1, 2) \to (2, 0) \to (0, 1) \to (2, 2) \to (0, 3) \to (1, 1)$. *Inductive step:* Assume the inductive hypothesis, that the knight can reach any square (i, j) for which $i + j = k$, where k is an integer greater than 1. We must show how the knight can reach each square (i, j) when $i + j = k + 1$. Because $k + 1 \ge 3$, at least one of i and j is at least 2. If $i \ge 2$, then by the inductive hypothesis, there is a sequence of moves ending at $(i - 2, j + 1)$, because $i - 2 + j + 1 = i + j - 1 = k$; from there it is just one step to (i, j); similarly, if $j \ge 2$. **57.** *Basis step:* The base cases $n = 0$ and $n = 1$ are true because the derivative of x^0 is 0 and the derivative of $x^1 = x$ is 1. *Inductive step:* Using the product rule, the inductive hypothesis, and the basis step shows that $\frac{d}{dx}x^{k+1} = \frac{d}{dx}(x \cdot x^k) = x \cdot \frac{d}{dx}x^k + x^k \frac{d}{dx}x = x \cdot kx^{k-1} + x^k \cdot 1 = kx^k + x^k = (k+1)x^k$. **59.** *Basis step:* For $k = 0$, $1 \equiv 1 \pmod{m}$. *Inductive step:* Suppose that $a \equiv b \pmod{m}$ and $a^k \equiv b^k \pmod{m}$; we must show that $a^{k+1} \equiv b^{k+1} \pmod{m}$. By Theorem 5 from Section 4.1, $a \cdot a^k \equiv b \cdot b^k \pmod{m}$, which by defini-