1. Numbers

There are different sets of numbers that we consider in this class:

a) The set of natural numbers, \( \mathbb{N} \)

\[ \mathbb{N} = \{ 1, 2, \ldots \} \]

Properties:
- The sum of two natural numbers is a natural number.
- The product of two natural numbers is a natural number.

Not always true:
- The difference between two natural numbers may not be a natural number.
- The division of two natural numbers may not be a natural number.

b) The set of integers, \( \mathbb{Z} \)

\[ \mathbb{Z} = \{ \ldots, -2, -1, 0, 1, 2, \ldots \} \]

\( \mathbb{N} \subset \mathbb{Z} \).
Properties:
- Sum of 2 integers is an integer.
- Product of 2 integers is an integer.
- Difference of 2 integers is an integer.

Not always true:
- The division of two integers may not be an integer.

b) The set of rational numbers \( \mathbb{Q} \)

A number \( x \) is a rational number if and only if there exists a pair of integers \((a, b)\) with \( b \neq 0 \) such that \( x = \frac{a}{b} \).

\((a, b)\) is not unique. In fact, for all \( k \) integer \( \neq 0 \), \( \frac{a}{b} = \frac{ak}{bk} \).

Properties:
- Sum of 2 rational numbers is a rational number.
- Difference of 2 rational numbers is a rational number.
- The product of 2 rational numbers is a rational number.
- The division of 2 rational numbers is a rational number as long as the divisor is not 0.
d) The set of real numbers \( \mathbb{R} \) is in \( \mathbb{C} \subset \mathbb{Z} \subset \mathbb{R} \subset \mathbb{N} \).

Solve \( x^2 = 2 \):

- If \( x \in \mathbb{N} \) no solution
- If \( x \in \mathbb{Z} \) \( \sqrt{2} \)
- If \( x \in \mathbb{Q} \) \( \sqrt{2} \)
- If \( x \in \mathbb{R} \), \( [- \sqrt{2}, \sqrt{2}] \)
Induction: To prove that 
\[ \forall n \in \mathbb{N}, n \geq m_0, \ P(n) \text{ is true.} \]

We show:

i) \( P(m_0) \) is true (basis step)

ii) \( P(k) \implies P(k+1) \), \( \forall k \geq m_0 \) (inductive step)

The method of proof by induction allows you to conclude that \( P(n) \) is true, \( \forall n \geq m_0 \).

Warning: Let me try to show that "all houses have the same color.

\[ P(n): \text{n houses have the same color.} \]
\[ \forall n \in \mathbb{N}, P(n) \text{ is true.} \]

Basis step: \( P(1) \) is true; a single house has one color, its own.

Inductive step: \( P(k) \implies P(k+1) \)

Direct proof: \( P(k) \) is true. Every time I pick \( k \) houses, they have the same color.

\[ S = \{ H_1, \ldots, H_{k+1} \} \]
\[ S = \left| H_1, \ldots, H_{k+1} \right| \]

\[ S = \left| H_1, \ldots, H_{k} \right| \cup \left| H_{k+1} \right| \]

\( k \) houses: \( C_1 \)

\( S_1 = \left| H_1, \ldots, H_{k} \right| \)

\( S_2 = \left| H_{k+1} \right| \)

\( S_1 \cup S_2 = \left| H_1, \ldots, H_{k} \right| \)

All houses in \( S_1 \) and \( S_2 \) have the same color, described as \( C_1 \).

All houses in \( S \) have the color \( C = C_1 = C_2 \).

However: if \( k = 1 \), the intersection \( S_1 \cap S_2 \) does not exist.

If this intersection does not exist, I cannot conclude \( C_1 = C_2 \).
Problem:

Any stampage value $n$ can be made with only 5 cents stamps and 3 cents stamps, when $n \geq 8$.

$P(n)$: There exists two positive integers $(a, b)$ such that

$$n = 3a + 5b$$

We want to show:

$$\forall n \geq 8, \ P(n) \text{ is true.}$$

Basis step: $P(8)$: $8 = 3 + 5$

$a = 1, b = 1$.

Inductive step: $P(k) \rightarrow P(k+1), \ k \geq 8$

$P(k)$ is true: there exists $(a, b)$ positive such that

$$k = 3a + 5b$$

$$k+1 = 3a + 5b + 1$$

$$= 3a + 5b + 6 - 5$$

$$= 3(a + 2) + 5(b - 1)$$