A) Fibonacci's Number

\[ F_n \text{ are defined using:} \]

- **Basis step:** \( F_0 = 0; \quad F_1 = 1 \)
- **Inductive step:** \( F_n = F_{n-1} + F_{n-2} \)

**Example:**

Show that \( F_n \leq 2^n, \quad \forall n \geq 1 \)

**Proof by induction:**

Let us define:

- **LHS(n):** \( F_n \)
- **RHS(n):** \( 2^n \)

**P(n):** \( \text{LHS}(n) \leq \text{RHS}(n) \)

**Basis step:** \( n = 1 \)

\[ \text{LHS}(1) = F_1 = 1 \quad \text{LHS}(1) \leq \text{RHS}(1) \]
\[ \text{RHS}(1) = 2^1 = 2 \quad P(1) \text{ is true.} \]

**Inductive step:** I want to show \( P(k-1) \land P(k) \rightarrow P(k+1) \)
What I know:

$P(k-1)$ is true: LHS$(k-1) \leq$ RHS$(k-1)$

$F_{k-1} \leq 2^{k-1}$

What I want to show:

$P(k+1)$ is true

LHS$(k+1) = F_{k+1}$

RHS$(k+1) = 2^{k+1}$

$F_{k+1} = F_k + F_{k-1}$

$F_{k+1} \leq 2^{k-1} + 2^k$

Option 1:

$2^{k-1} + 2^k = 2^{k-1} (1 + 2)$

$= 2^{k-1} \times 3$

$\leq 2^{k-1} \times 4$

$\leq 2^{k-1} \times 2^2$

$\leq 2^{k+1}$
Option 2:

Comparing $2^k - 2^{k-1}$ and $2^k$

\[
2^k - 2^{k-1} = 2^{k-1} (2 - 1) = 2^{k-1}
\]

$2^k \geq 2^{k-1}$

$2^{k-1} \leq 2^k$

$2^{k-1} + 2^k \leq 2^{k+1}$

$F_{k+1} \leq 2^{k-1} + 2^k \leq 2^{k+1}$

Option 3:

$2^{m-1} \leq 2^n \quad \forall m \geq 1$

The method of proof by strong induction allows me to conclude that $P_n$ is true, $\forall n \geq 1$. 

II. Set theory.

Definition: A set is an unordered collection of objects.

Note: a) It is important to understand that the objects belong to a specified domain.

b) How do we specify a set?

3 possible representations:

- Roster
  \[ S = \{ 1, 3, 5, 7, 9, 11, 13, 15, 17, 19 \} \]

- Implicit representation
  \[ S = \{ 1, 3, 5, \ldots \} \quad (997, 999) \]

- Venn diagram
  \[ S = \{ n \in \mathbb{Z} \mid 0 \leq n \leq 1000 \text{ and } n \text{ is odd} \} \]
To write that an object $x$ in a domain $D$ belongs to a set $S$, we write $x \in D$ and $x \in S$.

To indicate that a sub-group of objects $A$ belongs to a set $B$, we write $A \subset B$.

Venn diagram:

- Let $D$ be the domain. Let $A$ and $B$ be two sets in $D$.
- The elements of $D$ that belong to both $A$ and $B$ are said to belong to the intersection of $A$ and $B$, written $A \cap B$. 

"element" 

"subset"