Discussion 5: Solutions

ECS 17 (Winter 2024)

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Exercise 1

Let p and q be two propositions. The proposition p NOR q is true when both p and q are false, and it is false otherwise. It is denoted $p \downarrow q$.

a) *Write down the truth table for $p \downarrow q$*

p	q	$p \downarrow q$
T	T	F
T	F	F
F	T	F
F	F	T

b) *Show that $p \downarrow q$ is logically equivalent to $\neg(p \vee q)$*

p	q	$p \downarrow q$	$p \vee q$	$\neg(p \vee q)$
T	T	F	T	F
T	F	F	T	F
F	T	F	T	F
F	F	T	F	T

Therefore $p \downarrow q$ is logically equivalent to $\neg(p \vee q)$

c) *Find a compound proposition logically equivalent to $p \wedge q$ using only the logical operator $\downarrow$.*

Note first that since $p \downarrow q \Leftrightarrow \neg(p \vee q)$, we get $p \downarrow p \Leftrightarrow \neg p$ and therefore we know how to express the negation operator using only the NOR operator.

Now, notice that:

$$\begin{aligned} p \downarrow q &\Leftrightarrow \neg(p \vee q) \\ &\Leftrightarrow \neg p \wedge \neg q \end{aligned}$$

where the second logical equivalence comes from de Morgan's law. Let $A \Leftrightarrow \neg p$, then $\neg A \Leftrightarrow p$. Similarly, Let $B \Leftrightarrow \neg q$, then $\neg B \Leftrightarrow q$. We get:

$$\neg A \downarrow \neg B \quad \Leftrightarrow \quad A \wedge B$$

We have seen that $\neg A \Leftrightarrow A \downarrow A$, and $\neg B \Leftrightarrow B \downarrow B$. We conclude that:

$$A \wedge B \Leftrightarrow (A \downarrow A) \downarrow (B \downarrow B)$$

We verify using a table of truth:

A	B	$A \wedge B$	$A \downarrow A$	$B \downarrow B$	$(A \downarrow A) \downarrow (B \downarrow B$
T	T	T	F	F	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	F	T	T	F

d) *Find a compound proposition logically equivalent to $p \to q$ using only the logical operator $\downarrow$.*

p	q	$p \downarrow p$	$(p \downarrow p) \downarrow q$	$((p \downarrow p) \downarrow q) \downarrow ((p \downarrow p) \downarrow q)$	$p \to q$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	F	T	T

Exercise 2

Let p, q, and r be three propositions. Prove or disprove the following statements using truth tables:

a) *$(p \vee \neg p) \wedge (q \vee r)$ is equivalent to $q \vee r$*

b) *$(p \wedge \neg p) \vee (q \wedge r)$ is equivalent to $q \wedge r$*

Let us use equivalence instead to prove the two statements:

p	q	r	$\neg p$	$p \vee \neg p$	$q \vee r$	$(p \vee \neg p) \wedge (q \vee r)$
T	T	T	F	T	T	T
T	T	F	F	T	T	T
T	F	T	F	T	T	T
T	F	F	F	T	F	F
F	T	T	T	T	T	T
F	T	F	T	T	T	T
F	F	T	T	T	T	T
F	F	F	T	T	F	F

p	q	r	$\neg p$	$p \wedge \neg p$	$q \wedge r$	$(p \wedge \neg p) \vee (q \wedge r)$
T	T	T	F	F	T	T
T	T	F	F	F	F	F
T	F	T	F	F	F	F
T	F	F	F	F	F	F
F	T	T	T	F	T	T
F	T	F	T	F	F	F
F	F	T	T	F	F	F
F	F	F	T	F	F	F

a)

$$\begin{aligned}(p \vee \neg p) \wedge (q \vee r) &\Leftrightarrow T \wedge (q \vee r)\\ &\Leftrightarrow q \vee r\end{aligned}$$

b)

$$\begin{aligned}(p \wedge \neg p) \vee (q \wedge r) &\Leftrightarrow F \vee (q \wedge r)\\ &\Leftrightarrow q \wedge r\end{aligned}$$

Exercise 3

Let p and q be two propositions. Check if the proposition $B = \neg(p \rightarrow \neg q) \rightarrow \neg(p \leftrightarrow \neg q)$ is a tautology, a contradiction, or neither.

Let us build the truth table for B:

p	q	$\neg q$	$(p \rightarrow \neg q)$	$\neg(p \rightarrow \neg q)$	$p \leftrightarrow \neg q$	$\neg(p \leftrightarrow \neg q)$	B
T	T	F	F	T	F	T	T
T	F	T	T	F	T	F	T
F	T	F	T	F	T	F	T
F	F	T	T	F	F	T	T

From Column 8, we can conclude that B is a tautology.

Line	A	B	C	B says	C says	Compatibility
1	Knight	Knight	Knight	F	F	No: B would be a Knight who lies
2	Knight	Knight	Knave	T	F	Yes
3	Knight	Knave	Knight	T	F	No, B would be a Knave who tells the truth
4	Knight	Knave	Knave	F	F	Yes
5	Knave	Knight	Knight	T	F	No, C would be a Knight who lies
6	Knave	Knight	Knave	F	F	No, B would be a Knight who lies
7	Knave	Knave	Knight	F	F	No, C would be a Knight who lies
8	Knave	Knave	Knave	F	T	No, C would be a Knave who tells the truth

Exercise 4

Let us play a logical game. You find yourself in front of three rooms whose doors are closed. You are told that behind one of the doors there is a princess and behind the two others doors there are tigers. There is one guardian in front of each door. The guardian in front of door 1 tells you, "The princess is behind my door". The guardian in front of door 2 tells you, "There is exactly one liar among us and the princess is behind my door". The guardian in front of door 3 tells you, "We are all liars". Knowing that the guardians either always tell the truth, or always lie, can you say where the princess is? Justify your answer.

Let A, B, and C be the guardians in front of door 1, 2, and 3, respectively, and let us build the table for the possible options for A, B, and C, with no consideration about the princess yet. We will call a guardian a "Knight" if she always tells the truth, and "Knave" otherwise. We then check the validity of the (part of the) two statements from B and C that do not depend on the princess, and check the consistency of the truth values for those statements with the nature of B and C. Finally, we take into account what is said about the princess.

Therefore A is a Knight, B is a Knight or a Knave, and C is a Knave. Note that A and B cannot be both Knights, as both say that they guard the door with the princess behind. Therefore, A is a Knight, B is a Knave, and C is a Knave. Since A is a Knight, the princess is behind door 1.