Data, Logic, and Computing

ECS 17 (Winter 2025)

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Discussion 9: Induction

Exercise 1

Prove by induction that every number greater that 7 is the sum of a nonnegative integer multiple of 3 and a nonnegative integer multiple of 5.

Let us rewrite this as:

$P(n)$: For all $n \geq 8$, there exist two integers $k \geq 0$ and $l \geq 0$ such that $n = 3k + 5l$. We prove it by induction.

- *basis step*: For $n = 8$, we can set $k = 1$ and $l = 1$: $8 = 3 * 1 + 5 * 1$. $P(8)$ is true.
- *Inductive step*. Let $P(n)$ be true for an integer $n \geq 8$, i.e. there exist $k \geq 0$ and $l \geq 0$ such that $n = 3 * k + 5 * l$. We want to decompose $n + 1$.

  Notice that: $n+1 = 3*k+5*l+1 = 3*(k+2)-6+5*(l-1)+5+1 = 3*(k+2)+5*(l-1)$. $k + 2$ is nonnegative; however, $l - 1$ may be negative if $l = 0$. We study two cases:

  - $l = 0$ then $n = 3k$, i.e. n is a multiple of 3. Since n is greater than 7, $k \geq 3$.
    Notice that $n + 1 = 3k + 1 = 3(k - 3) + 2 * 5$. Since $k \geq 3$, $k - 3 \geq 0$: we have found two nonnegative integers $m = k - 3$ and $p = 2$ such that $n + 1 = 3m + 5p$.
  - $l > 0$ then $l - 1 \geq 0$. Therefore we have found two nonnegative integers $m = k + 2$ and $n = l - 1$ such that $n + 1 = 3m + 5p$.

  In all cases, $P(n + 1)$ is true.

According to the principle of mathematical induction, we can conclude that for all $n \geq 8$, there exist two integers $k \geq 0$ and $l \geq 0$ such that $n = 3k + 5l$.

Exercise 2

Prove by induction that $2^{n+1} > n^2 + 1$ for all $n \geq 2$.
Let us define $LHS(n) = 2^{n+1}$ and $RHS(n) = n^2 + 1$.

- *Basis case:* Let us prove that $P(2)$ is true:
  $LHS(2) = 2^3 = 8$
  $RHS(2) = 2^2 + 1 = 5$.
  Therefore $LHS(2) > RHS(2)$. P(2) is true.

- *Inductive step:* Let us assume that $P(n)$ is true. This means that $LHS(n) > RHS(n)$. We want to prove $P(n+1) : LHS(n+1) > RHS(n+1)$ with $LHS(n+1) = 2^{n+2}$ and $RHS(n+1) = (n+1)^2 + 1$.

$$\begin{aligned} LHS(n+1) &= 2^{n+2} \\ &= 2 \times 2^{n+1} \\ &= 2 \times LHS(n) \\ &> 2 \times RHS(n) \\ &> 2(n^2+1) \end{aligned}$$

  Let us rewrite:

$$\begin{aligned} 2(n^2+1) &= n^2 + n^2 + 2 \\ &= n^2 + 2n + 2 + n^2 - 2n \\ &= (n+1)^2 + 1 + n^2 - 2n \\ &= (n+1)^2 + 1 + n(n-2) \end{aligned}$$

  Since $n \geq 2$, $n(n-2) \geq 0$, therefore $2(n^2+1) \geq (n+1)^2 + 1$.
  Replacing above, we get:

$$\begin{aligned} LHS(n+1) &> (n+1)^2 + 1 \\ &> RHS(n+1) \end{aligned}$$

  Therefore $P(n+1)$ is true.

According to the principle of mathematical induction, we can conclude that $P(n)$ is true for all $n \geq 2$.

Exercise 3

Show that $1 + 3 + \ldots 2n - 1 = n^2$, for all $n \geq 1$.

Let us define $LHS(n) = 1 + 3 + \ldots 2n - 1$
and $RHS(n) = n^2$
Let $p(n) : LHS(n) = RHS(n)$
We want to show $p(n)$ is true for all $n \geq 1$

a) Base Case n=1
   $LHS(1) = 1$
   $RHS(1) = 1^2 = 1$
   Since $LHS(1) = RHS(1)$, $p(1)$ is true

b) Inductive Step
I want to show p(k)→p(k+1) whenever k≥1
Hypothesis: p(k) is true and LHS(k)=RHS(k)

$$\begin{aligned} LHS(k+1) &= 1+3+\ldots 2n-1+2n+1 \\ &= LHS(k)+2n+1 \\ &= RHS(k)+2n+1 \\ &= n^2+2n+1 \\ &= (n+1)^2 \\ &= RHS(k+1) \end{aligned}$$

Therefore $LHS(k+1) = RHS(k+1)$, which validates that $P(k+1)$ is true.

The principle of proof by mathematical induction allows us to conclude that $P(n)$ is true for all $n \geq 1$.

Exercise 4

Show that $\displaystyle\sum_{k=1}^{n} \frac{1}{4k^2-1} = \frac{n}{2n+1}$ *for all integer* $n \geq 1$.

Let us define $LHS(n) = \sum_{k=1}^{n} \frac{1}{4k^2-1}$
and $RHS(n) = \frac{n}{2n+1}$
Let $p(n) : LHS(n) = RHS(n)$
We want to show $p(n)$ is true for all $n \geq 1$

a) Base Case n=1
$LHS(1) = \frac{1}{3}$
$RHS(1) = \frac{1}{2\times1+1} = \frac{1}{3}$
Since $LHS(1) = RHS(1)$, $p(1)$ is true

b) Inductive Step
I want to show p(k)→p(k+1) whenever k≥1
Hypothesis: p(k) is true and LHS(k)=RHS(k)

$$
\begin{aligned}
LHS(k+1) &= \sum_{i=1}^{k+1} \frac{1}{4i^2-1} \\
&= LHS(k) + \frac{1}{4(k+1)^2-1} \\
&= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} \\
&= \frac{k(2k+3)+1}{(2k+1)(2k+3)} \\
&= \frac{2k^2+3k+1}{(2k+1)(2k+3)} \\
&= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)} \\
&= \frac{k+1}{2k+3} \\
&= RHS(k+1)
\end{aligned}
$$

Therefore $LHS(k+1) = RHS(k+1)$, which validates that $P(k+1)$ is true.

The principle of proof by mathematical induction allows us to conclude that $P(n)$ is true for all $n \geq 1$.

Exercise 5

Let f_n be the n-th Fibonacci number. Show that f_{3n} is even for all $n \geq 1$.

We note first that all Fibonacci numbers are integers. We use a proof by induction.

- *Basis step:* $f_3 = f_2 + f_1 = 1 + 1 = 2$ which is even. The proposition is true for $n = 1$.
- *Inductive step:*. We suppose that f_{3n} is even for $n \geq 1$. Then:
  $f_{3n+3} = f_{3n+2} + f_{3n+1} = f_{3n+1} + f_{3n} + f_{3n+1} = 2 \times f_{3n+1} + f_{3n}$.
  $2 \times f_{3n+1}$ is even and f_{3n} is even by hypothesis; therefore $f_{3(n+1)}$ is even.

According to the principle of mathematical induction, we can conclude that f_{3n} is even for all $n \geq 1$.

Exercise 6

Let f_n be the n-th Fibonacci number (note: Fibonacci numbers satisfy $f_0 = 0$, $f_1 = 1$ and $f_{n+2} = f_{n+1} + f_n$). Let m be a fixed strictly positive integer. Prove by strong induction that for all $n \geq 0$, $f_{n+m} = f_m f_{n+1} + f_{m-1} f_n$.

We define $LHS(n) = f_{n+m}$ and $RHS(n) = f_m f_{n+1} + f_{m-1} f_n$. Let $p(n)$ be the proposition: $LHS(n) = RHS(n)$.

We want to show that $p(n)$ is true for all $n \geq 0$.

a) Base Case n=0
$LHS(0) = f_m$
$RHS(0) = f_m f_1 + f_{m-1} f_0 = f_m$
Therefore $LHS(0) = RHS(0)$ and $p(0)$ is true.

b) Inductive Step
I want to show $[p(0) \wedge p(1) \wedge \ldots \wedge p(k)] \rightarrow p(k+1)$ whenever $k \geq 1$

We note first that:
$LHS(k+1) = f_{k+1+m}$
$RHS(k+1) = f_m f_{k+2} + f_{m-1} f_{k+1}$

From the definition of Fibonacci numbers,
$LHS(k+1) = f_{k+1+m} = f_{k+m} + f_{k-1+m}$
Since $p(k)$ is true, $f_{k+m} = f_m f_{k+1} + f_{m-1} f_k$. Similarly, since $p(k-1)$ is true, $f_{k-1+m} = f_m f_k + f_{m-1} f_{k-1}$. Replacing in the equation above, we get:

$$\begin{aligned} LHS(k+1) &= f_m f_{k+1} + f_{m-1} f_k + f_m f_k + f_{m-1} f_{k-1} \\ &= f_m (f_{k+1} + f_k) + f_{m-1}(f_k + f_{k-1}) \\ &= f_m f_{k+2} + f_{m-1} f_{k+1} \\ &= RHS(k+1) \end{aligned}$$

Therefore $LHS(k+1) = RHS(k+1)$ which validates that $p(k+1)$ is true.

The principle of proof by mathematical induction allows us to conclude that $p(n)$ is true for all $n \geq 0$.